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HISTORY

Thureau-Dangin, F. Sketch of a history of the sexagesimal system. Osiris 7, 95–141 (1939). [MF 424]

This paper is a considerably enlarged English edition of a short French book on the same subject, published in 1932. The author treats the Sumerian number system, expressions for fractions, metrology and the division of the circle, and the systematic number and fractional expressions of the latest periods. Examples from mathematical and astronomical texts are given and relations to the later development are indicated. Many bibliographical references.

O. Neugebauer (Providence, R. I.).

Heidel, W. A. The Pythagoreans and Greek mathematics. Amer. J. Philology 61, 1–33 (1940). [MF 1489]

This critical study of the tradition of "Pythagorean" mathematics concludes with the statement "that it is impossible to reconstruct the history of Greek mathematics, as one may to a certain extent tell the story of the development of Greek scientific thought in general, by focusing attention upon individual men or groups. Regarding our knowledge of details and also with respect to the necessary inferences from known facts nothing is changed; but the rôle of the Pythagoreans must appear to have been much exaggerated." *O. Neugebauer* (Providence, R. I.).

Reidemeister, Kurt. Die Arithmetik der Griechen. Hamburg Math. Einzelsch. **1939**, no. 26, 32 pp. (1940). [MF 1206]

The author investigates the reliability of the known reports on the earliest Greek mathematics and comes to the conclusion that they are very untrustworthy and a product of neo-Pythagorean speculations on Platonic or pseudo-Platonic dialogues. By discussing Euclid's Elements it is possible to isolate connected arithmetical theorems of dyadic character on the one hand and on the other the remains of an older arithmetical theory of geometrical series. These arithmetical theories are finally connected with the theory of musical harmonic intervals, without any assumption of the "scandale logique," initiated by the discovery of the irrational numbers. *O. Neugebauer* (Providence, R. I.).

Gandz, Solomon. Studies in Hebrew mathematics and astronomy. Proc. Amer. Acad. Jewish Res. **9**, 5–50 (1939). [MF 876]

The first part contains remarks on early Hebrew mathematics, for example, on quadratic equations and on the formula of the frustrum, which appears first in the Moscow papyrus and then again for the first time in medieval times in the work of Savasorda. The second part concerns the zodiacal light. *O. Neugebauer* (Providence, R. I.).

Zilsel, Edgar. Copernicus and mechanics. J. Hist. Ideas 1, 113–118 (1940). [MF 1488]

✶Rosen, Edward. Three Copernican Treatises. The Commentariolus of Copernicus. The Letter against Werner. The Narratio prima of Rheticus. Records of Civilization, Sources and Studies, no. 30. Columbia University Press, New York, 1939. x+211 pp. $3.00.

This book contains the first English translation of the two smaller treatises of Copernicus, mentioned in the title, and of the first report of Rheticus on the large work "De Revolutionibus," at this time under preparation. However, this book is much more than a simple translation. It represents practically a new edition, of the highest scholarly rank, of these famous works. In about 600 notes the readings in the different versions of the text are discussed, connections between the theory of Copernicus and his predecessors are indicated, and the calculations are explained. A large and very elaborate index enables the reader to find the treatment of the different problems both in the text and in the critical notes. The introduction contains a short survey of the character of the translated works and of the fundamental astronomical concepts used in the text together with a valuable discussion of Copernicus' terminology. In the preface the project of a first English translation of the "De Revolutionibus" is announced. No more promising beginning could be given to this enterprise than this introductory publication. *O. Neugebauer.*

✶Turnbull, Herbert Westren, ed. James Gregory Tercentenary Memorial Volume. Royal Society of Edinburgh. G. Bell and Sons, Ltd., London, 1939. xi+524 pp. 25s.

After giving a survey of Gregory's life and work and of his scientific surroundings, the largest part of the volume is dedicated to his extremely interesting correspondence, mainly with John Collins in 1667–1675. Most of Gregory's own letters have been edited in a collection by Rigaud [Oxford, 1841], which is, however, out of print. The letters of Collins were recently found. They give important disclosures about Gregory's discoveries and their relations to the work of other contemporary mathematicians. Turnbull has examined extremely carefully all this material and many related original papers; he presents the result in many elucidating notes. The letters and these notes cover almost all topics with which the mathematicians of that period were concerned. They show Gregory's genius and his mathematical intuition to have been still greater than one could guess by the facts known up to now.

Perusing the letters which were published formerly, Whittaker and Gibson had found out many years ago that Gregory had anticipated Newton in discovering both the interpolation formula and the general binomial theorem as early as 1670. The new material corroborates and completes these facts and shows that Gregory made important applications of these discoveries, deducing from them expansions of many important functions. Moreover, although proofs are not given, the succession of the results makes it very probable that he found the binomial theorem by applying

his interpolation formula to the expression $b(1+b/d)^{a/c}$ as a function of a/c [p. 131 ff.]. Shortly afterwards he found a lot of new series, which he communicated to Collins in February, 1671 [tg x, sec x, log sec x, etc., p. 170 ff.]. Turnbull shows from Gregory's notes that just at that time he calculated the successive derivatives of those functions and that he used them to form the series is almost certain, because not only the coefficients of the series are exactly those found in the notes, but the only numerical error which is contained in the series occurs in the notes too [p. 350 ff.]. So Gregory was using Taylor expansions more than 40 years in advance of Taylor. Unfortunately he tells in the letters only applications of this formula, while he stated the interpolation formula in general form. He seems to have supposed at this time that Newton had reached this formula too, and he did not publish any of his great discoveries in the few years he had to live. Among other problems of analysis treated by Gregory are: (1) He solves Kepler's famous problem (to divide a semicircle by a straight line through a given point of the diameter in a given ratio) by means of a series which he obviously finds by applying the Taylor series to the general cycloid [p. 226 ff., p. 363]. (2) He gives one of the earliest examples of a comparison test for convergence, virtually Cauchy's ratio test, together with an appraisal of the remainder [p. 230]. The exactness of his proofs sometimes looks almost modern. In one of the notes [p. 445] he gives a definition of the integral which is virtually as general as Riemann's notion. (3) He proposes to solve the differential equation $y^2(1+y'^2)=f(x)$, especially for $f(x)=\text{const}$. Barrow gives its singular solution in a letter, published in this volume and followed by remarks of Gregory about all possible solutions; this discussion is very impressive because of the acuteness of thought of both men. (4) Gregory approaches in some cases the problem of proving that there is no "analytic proportion" (that is, in modern language, algebraic equation) between certain geometrical quantities [p. 96, 102]; it is known that he was the first who tried to prove the non-analytic character in that sense of a circular or hyperbolic sector, in his "Vera Quadratura" [p. 468 ff.].

Besides analysis, problems from other ranges are occasionally treated. For instance, in algebra Gregory was able to express the sum of the nth powers of the roots in terms of the coefficients [p. 211, 302 ff.], here a predecessor of Newton too; and he was concerned with methods for solving higher algebraic equations by radicals, but a remark in his last letter to Collins [p. 335] could perhaps indicate that he began to realize the futility of these efforts. Also theory of numbers is treated in some notes [p. 398 ff.]; he investigates some Diophantine equations of Fermat, whose work he probably knew, and some related problems obviously in an independent way and with some results, new at that time. Also other interests of Gregory are covered in the correspondence, especially his astronomical work; some letters of his and of Newton deal with the invention of the telescope, in which again their work overlapped.

The last part of the volume contains essays on the published books of Gregory, written for the tercentenary by various authors. They treat the Optica Promota and the Exercitationes Geometricae (H. W. Turnbull), the Vera Quadratura (M. Dehn and E. Hellinger) [cf. these Rev. 1, 33 (1940)] and the Geometriae Pars Universalis (A. Prag), further Gregory's sojourn in Italy (F. Enriques) and the relations between him and Huygens (E. J. Dijksterhuis).

E. Hellinger (Evanston, Ill.).

Carruccio, Ettore. L'estrazione di radice cubica, mediante inserzione di due medie proporzionali fra due segmenti dati, in Leonardo Pisano. Period. Mat. **19**, 189–197 (1939). [MF 915]

Bortolotti, Ettore. L'Opera geometrica di Evangelista Torricelli. Monatsh. Math. Phys. **48**, 457–486 (1939). [MF 662]

This article contains a concise survey of the geometrical investigations of Evangelista Torricelli, in which his theorems and demonstrations are exposed in modern symbolism as well as in the original form. An essential quality of his mathematical work consists in his aim at general methods. For example, he gives a general demonstration of the theorem of the calculus which is expressed nowadays by the formula $\int x^n dx=(1/(n+1))\cdot x^{n+1}$ ($n\neq -1$, rational). By a close examination of the nature of Cavalieri's indivisibles he is led to a solution of the problem of the tangent in the case of the generalized parabolas and hyperbolas which are determined by the equation $y^n=kx^m$ (m and n integers). It is a noteworthy discovery that the problems of the quadrature, the tangent and the centre of gravity of these curves all depend on one and the same differential equation. Besides the definite integral which is incorporated in the solution of the quadrature, Torricelli makes use of the indefinite integral which is applied in his theory of motion. Substantially he is acquainted with the inverse character of the operations of definite integration and differentiation. By introducing the concept of curved indivisibles he considerably extends the domain of the transformations used in the evaluation of integrals; moreover, he prepares the modern curvilinear coordinates. He applies his method of integration to the problem of rectification of plane curves; in particular, he determines the total length of the so-called geometrical (that is, logarithmical) spiral. The theory of the centre of gravity of geometrical figures is promoted by his penetrating investigations; he is the first to obtain the coordinates of this centre as a quotient of two definite integrals. According to the author the merits of Torricelli are greatly underrated. *E. J. Dijksterhuis* (Oisterwijk).

Berzolari, Luigi. Gaetano Scorza. Boll. Un. Mat. Ital. (2) **1**, 401–408 (1939). [MF 1309]

Hjelmslev, Johannes. Hieronymus Georg Zeuthen. Address given before the Matematisk Forening on the occasion of the celebration of the 100th birthday of H. G. Zeuthen. Mat. Tidsskr. A **1939**, 1–10 (1939). (Danish) [MF 730]

Chatley, Herbert. Ancient Chinese astronomy. Occasional Notes. Roy. Astr. Soc. **1939**, 65–74 (1939). 1 plate. [MF 877]

Report on the development of Chinese astronomy.

O. Neugebauer (Providence, R. I.).

Fujiwara, M. Miscellaneous notes on the history of Wazan, the old Japanese mathematics. I and II. Tôhoku Math. J. **46**, 123–144 (1939). (Japanese) [MF 1175]

Dittrich, A. Die Finsternistafel des Dresdener Maya-Kodex. Abh. Preuss. Akad. Wiss. **1939**, no. 2, 47 pp. (1939). [MF 997]

FOUNDATIONS

★Gonseth, F. Philosophie mathématique. Actual. Sci. Ind. 837. Philosophie. Chronique annuelle. IV. Hermann & Cie, Paris, 1939. 100 pp.

This is a philosophical discussion of some questions concerning the epistemological foundations of mathematics. It is part of a philosophical series and its character is more philosophical than it is mathematical. The reviewer found it obscure and difficult reading. The primary object is to expound the philosophical background of work done in the foundations of mathematics during the years 1937–38; but most of the space is devoted to more general considerations.

H. B. Curry (State College, Pa.).

Aida, Gundayu. Tensors and universalism. Tensor 2, 46–48 (1939). (Japanese) [MF 1370]

Some philosophical discussions on the relation of the universalism of the ideas in tensor calculus and physics.

A. Kawaguchi (Sapporo).

Dehn, Max. Ueber Ornamentik. Norsk Mat. Tidsskr. 21, 121–153 (1939). [MF 869]

★Woodger, J. H. The Technique of Theory Construction. International Encyclopedia of Unified Science, vol. 2, no. 5. University of Chicago Press, Chicago, 1939. vii+81 pp. $1.00.

The author maintains that what is common to all sciences is mathematical logic. If this is true, the unity of science is best realized by formalizing all scientific theories in a common notation, that of symbolic logic. Different theories will then differ only in the subject matter constants specific to them. Such logical formalization, besides correcting some undesirable effects of specialization, will aid individual sciences by clearing up confusions due to the inexactness of common speech, and by making it easier to discover what are the consequences of new hypotheses to be tested.

To illustrate the technique required, a sample of such a formalized theory is given in great detail, somewhat along the lines of the same author's "Axiomatic Method in Biology" [Cambridge, 1937], using a symbolism like that of Principia Mathematica. The distinctions between syntax and semantics, theory and metatheory, postulates and rules of procedure, emphasized by Carnap and Tarski, are made clear. In addition to the familiar logical constants, the sample theory contains three specific subject matter constants, the relations $x\mathbf{P}y$ (x is part of y), $x\mathbf{T}y$ (x precedes y in time), and the class **cell** of cells in the sense of biology. A few simple assumptions lead to theorems such as "no cell arises both by division and fusion." In answer to the objection that the biological sciences are by tradition inductive and not deductive, the author suggests that this tradition may be one reason why progress in these sciences has been less rapid than in the physical sciences.

O. Frink (State College, Pa.).

Lietzmann, W. Über das Verhältnis von Definition und Lehrsatzgefüge. Monatsh. Math. Phys. 48, 141–146 (1939). [MF 633]

The author makes some general remarks concerning the role of definitions in the presentation of a mathematical theory, including the psychological factors bearing on the choice of suitable definitions. He illustrates his remarks by illustrations drawn mainly from the theory of conic sections.

H. B. Curry (State College, Pa.).

McKinsey, J. C. C. Proof of the independence of the primitive symbols of Heyting's calculus of propositions. J. Symbolic Logic 4, 155–158 (1939). [MF 747]

To show that the formula $\neg a \supset S . \wedge . S \supset \neg a$, where S does not contain $\neg$, cannot be provable, the author constructs a matrix in which (1) the value of every provable formula is always the "designated" element and (2) a proper subclass is closed with respect to $\wedge, \vee, \supset$, but not with respect to $\neg$. The same method applies, respectively, to $\wedge, \vee, \supset$ in the place of $\neg$.

A. Heyting (Laren).

Wernick, William. An enumeration of logical functions. Bull. Amer. Math. Soc. 45, 885–887 (1939). [MF 775]

The author considers functions of n variables, where the variables as well as the functional values range over a finite set of m values, which may be taken as the m truth values of a many valued logic. He shows that the number of such functions which depend on exactly k of the variables, but are independent of the other $n-k$, is given by the formula

$$C(n, k) \sum_{i=0}^{k} (-1)^i C(k, i) m^{m^{k-i}}.$$

O. Frink (State College, Pa.).

Fitch, Frederic B. Note on modal functions. J. Symbolic Logic 4, 115–116 (1939). [MF 465]

Some remarks and theorems extending the author's previous method for representing modal functions in two-valued logic [J. Symbolic Logic 2, 125–128 (1937)]. A Boolean representation of Lewis's system of strict implication is obtained wherein a prime non-zero element t is introduced, such that, while t is not itself tautological, yet it strictly implies all true propositions, and also is implied only by itself and zero.

A. A. Bennett (Providence, R. I.).

Fitch, Frederic B. The hypothesis that infinite classes are similar. J. Symbolic Logic 4, 159–162 (1939). [MF 748]

If the hypothesis that all infinite classes have the same cardinal number be called "the hypothesis of similarity," the author shows that "the system of logic of Whitehead and Russell's Principia Mathematica (without the axiom of reducibility) is consistent when to its axioms are added all the following: (1) the hypothesis of similarity, (2) the axiom of infinity, (3) the axiom of choice, (4) the contradictory of the axiom of reducibility."

B. A. Bernstein.

Parry, William Tuthill. Modalities in the *Survey* system of strict implication. J. Symbolic Logic 4, 137–154 (1939). [MF 746]

Functions of a single variable formed from the operations $\sim p$ (p is false) and $\Diamond p$ (p is possible) alone are called modalities. The system of strict implication S2 now preferred by C. I. Lewis, which arises when the postulate $\Diamond(pq) \prec \Diamond p$ is added to the system S1 of postulates B1–B7 of Lewis and Langford's Symbolic Logic, presumably has infinitely many distinct modalities. In the present paper it is shown that in the earlier system S3 of Lewis' Survey of Symbolic Logic, obtained by adding the stronger postulate $p \prec q \cdot \prec \cdot \sim\Diamond q \prec \sim\Diamond p$ to S1, there are 42 distinct modalities to which all others are reducible. The most complicated of these is $\sim\Diamond\sim\Diamond\sim\Diamond\Diamond\sim\Diamond\sim p$. All implications between these 42 are listed.

The system S4 of Becker obtained by adding $\sim\Diamond\sim p \rightarrow\!\!\!\!\!\!\;\; \sim\Diamond\Diamond\sim p$ to S1 is shown to have 14 distinct modalities. It is shown that adding the postulates $\sim\Diamond\sim\Diamond\Diamond p \rightarrow\!\!\!\!\!\!\;\; \sim\Diamond\sim\Diamond p$ and $\sim\Diamond\sim\Diamond\sim\Diamond p \rightarrow\!\!\!\!\!\!\;\; \sim\Diamond p$, respectively, to S3 gives rise to two new systems intermediate between S3 and S4, each having 26 modalities. Various postulates are shown to lead to the 6-modality system S5 of Becker. A system S4.5, which presumably has 10 modalities and is intermediate between S4 and S5, is discussed, but it is not shown to be distinct from S5. The author also gives an example of an 8-modality system, and of a system which he believes to have 12 modalities. It is shown that the postulate $\sim\Diamond\sim p = \sim\Diamond\sim\Diamond p$ of Becker reduces S1 to the system of material implication, contrary to a surmise of Churchman. The great amount of detail which the author gives will undoubtedly help in answering other questions concerning modalities in different systems of strict implication.

O. Frink (State College, Pa.).

Kalmár, László. On the possibility of definition by recursion. Acta Litt. Sci. Szeged **9**, 227–232 (1940). [MF 1226]

In the arithmetic based on Peano's axioms, sum, product and power are defined by recursion equations of the type (1) $\phi(0)=\alpha$, $\phi(n')=\beta(n, \phi(n))$. It is necessary to prove the existence of a function ϕ satisfying (1) for all n. Dedekind's existence proof requires defining the relation $m \leqq n$ in a complicated manner independent of the arithmetical operations. Lorenzen [Monatsh. Math. Phys. **47**, 356–358 (1939)] has given a simpler existence proof using multivalent functions. The author gives a still simpler proof using the idea of a partial solution of (1). By this is meant a function $\psi(n)$ such that, if $\psi(n')$ is defined, $\psi(n)$ is defined also and $\psi(n')=\beta(n, \psi(n))$. He shows by induction that for each n a partial solution exists for which $\psi(n)$ is defined, and that two partial solutions have the same values when both are defined. The existence of a $\phi(n)$ satisfying (1) follows.

O. Frink (State College, Pa.).

Burckhardt, Johann Jakob. Zur Neubegründung der Mengenlehre. Folge. Jber. Deutsch. Math. Verein. **49**, 146–155 (1939). [MF 684]

This is the second part of the author's exposition of the Finsler system of set theory; the first part appeared in Jber. Deutsch. Math. Verein. **48**, 146–165 (1938). This second part deals with the definition of the number series and some related concepts within the system. The author also discusses the objection to the system originally made by Baer, but the answer given by the author does not seem at all satisfactory.

H. B. Curry (State College, Pa.).

Glagoleff, Nil. Sur les axiomes d'appartenance de la géométrie euclidienne. Rec. Math. (Moscou) **6** (48), 221–225 (1939). (French. Russian summary) [MF 1355]

The author has previously shown [Period. Mat. **14**, 172–176 (1935)] how the axioms of connection of euclidean geometry and the axioms of projective geometry may be simplified by introducing two incidence axioms (which we express in the convenient "on" language): (a) If an element A is on an element B, then B is on A. (b) If the point A and the plane α are on the line a, then they are on one another. In the present note new consequences of these axioms are obtained.

Adjoin to (a) and (b) the following axioms: I. There exists a single line on two given distinct points. II. If two different planes α, β are on a point A, they are on a line which is on α, β and the point A. III. If α is a plane there exists at least one point on α. IV. There is one and only one plane α which is on a point A and a line a not on A. In the plane α there is one and only one line b that is on A and has no point on line a. V. There are at least four points not on any plane or any line. Then the system (a), (b), I–V forms an independent set of axioms of connection for euclidean geometry. The following theorems, which appear as axioms in Hilbert's "Grundlagen" [7te Aufl.] are proved: (1) There are at least three points not on any line. (2) Corresponding to each set of three points not on a line there is one and only one plane which is on the points. (3) If two points A, B are on a plane α and a line a, then the line a is on α. (4) On any line there are at least two points. This last theorem is noteworthy since, a priori, no assumption is made that any line contains even one point.

L. M. Blumenthal (Columbia, Mo.).

Köthe, Gottfried. Unendliche Abelsche Gruppen und Grundlagen der Geometrie. Jber. Deutsch. Math. Verein. **49**, 97–113 (1939). [MF 678]

This is an expository report primarily on the characterization of the fundamental fields of geometry (real, complex, quaternion, Cayley) as particularized topological Abelian groups. The essential group and topological definitions are given, a number of the simpler theorems, those mainly which deal with character groups, are proved and the leading ideas of more intricate arguments are referred to a very adequate bibliography.

L. Zippin (New York, N. Y.).

Ogasawara, Tôzirô. Relation between intuitionistic logic and lattice. J. Sci. Hirosima Univ. Ser. A **9**, 157–164 (1939). [MF 670]

It is proved that the intuitionistic propositional calculus as formulated by A. Tarski [Fund. Math. **31**, 103–104 (1939)] may be interpreted as a residuated distributive lattice, the residuation being with respect to the cross-cut operation in the lattice [M. Ward, Ann. of Math. **39**, 558–568 (1938)].

M. Ward (Pasadena, Calif.).

Skolem, Th. Eine Bemerkung über die Induktionsschemata in der rekursiven Zahlentheorie. Monatsh. Math. Phys. **48**, 268–276 (1939). [MF 644]

In addition to the rule of procedure for mathematical induction (namely, that from $A(0)$, and $A(a) \rightarrow A(a')$, follows $A(a)$), Hilbert and Bernays in their treatment of recursive number theory [Grundlagen der Mathematik, Bd. 1] give three extended induction schemas for functions of two variables. The first of these is that from $A(b, 0)$, and $A(\phi(b, a), a) \rightarrow A(b, a')$, follows $A(b, a)$. The author shows that these three more complicated induction schemas are consequences of the schema for simple mathematical induction. To make the reduction, however, it is necessary to introduce new functions (of three variables) which, though recursively defined, are not primitive recursive functions. The author remarks that probably all such higher induction schemas are reducible to simple induction. Their use is justifiable, however, since they lead to shorter proofs.

O. Frink (State College, Pa.).

Péter, Rózsa. Contribution to recursive number theory. Acta Litt. Sci. Szeged **9**, 233–238 (1940). [MF 1227]

In the paper reviewed above, Skolem has shown that three rules of procedure which are generalizations of simple mathematical induction, and are used by Hilbert and Ber-

nays in their treatment of recursive number theory, are reducible to simple mathematical induction. Skolem's proof requires the introduction of functions defined by non-primitive recursion equations of the form (1) $\phi(n, a)=\alpha(a)$, $\phi(n', a)=\beta(n, a, \phi(n, \gamma(n, a)))$, where α, β, and γ are primitively recursive functions. The author shows that, contrary to a surmise of Skolem, a function ϕ defined by (1) may also be defined by a succession of primitive recursions and substitutions. Hence Skolem's proof may be modified so as to use only primitive recursions. In this paper the existence of a function satisfying (1) is proved, whereas in her previous work on the reduction of recursions to primitive recursions [Math. Ann. 110, 612–632 (1934)] this existence was assumed. *O. Frink* (State College, Pa.).

THEORY OF NUMBERS

★**Lehmer, D. N. Factor Stencils.** Revised and extended by John D. Elder. Carnegie Institution of Washington, Washington, 1939. 27 pp.+2135 stencils. (Not for sale.)

This new edition of the Factor Stencils has only the basic idea in common with the first edition which appeared in 1929. The new stencils are punched Hollerith cards. For each square free number R less than 250 in absolute value, there are seven stencils each with 800 cells corresponding to the first 5600 primes ($\leqq 55079$). Those cells are punched out whose corresponding primes divide R or admit R as a quadratic residue.

Since every quadratic residue R of a number N is also a quadratic residue of every possible factor of N, it follows that the problem of factoring a number N is hereby reduced to the discovery of an adequate number of quadratic residues R of N and the superposition of the corresponding stencils to reveal those few primes having these residues R. The pamphlet which accompanies the stencils explains and illustrates various practical methods for the discovery of quadratic residues of a given large number N and differs from the original pamphlet only in the description of the stencils.

The new edition is an extension of the old both as to the number of residues R considered and also the number of primes treated. The old edition extended to $|R|<240$ and covered 5000 primes. The discovery of ten or a dozen residues of a number N will reveal all prime factors of N (if any) lying below 55103 and hence the stencils serve to examine completely any number $N<55103^2=3,036,340,609$. In the preparation of the new edition, the old edition was used only for comparison purposes, all computations, including the tables of linear divisors of the forms x^2-py^2, upon which the stencils are based, were done afresh. No error was found in the tables of linear divisors but about 1100 errors were found in the old stencils, "due almost entirely to the mechanical limitations" of the punching methods used originally. The automatic nature of modern card punching equipment is responsible for the remarkable fact only one error was detected in the master set for the new edition in the final comparison with the original edition, one error in 1708000 cells. The accuracy and rapidity with which the new stencils can be superimposed adds greatly to the efficiency of this factorization method. The new medium also assures the reliable and rapid reproduction of the stencils as long as I.B.M. equipment remains essentially unchanged.

Professor Archibald calls attention to the fact that the remarks in the historical introduction concerning Babylonian factorization tablets are incorrect.

D. H. Lehmer (Bethlehem, Pa.).

Kamber, F. Sur les suites de la forme $z=(2x+1)2^y+1$. Sphinx 9, 100–103 (1939). [MF 995]

Moessner, Alfred. Einige numerische Identitäten. Proc. Indian Acad. Sci., Sect. A. 10, 296–306 (1939). [MF 689]

The author considers the two systems of Diophantine equations

$$\sum_{i=1}^{3} G_i^n = \sum_{i=1}^{3} J_i^n, \qquad n=1, 3,$$

and

$$\sum_{i=1}^{3} K_i^n = \sum_{i=1}^{4} L_i^n, \qquad n=1, 3.$$

He exhibits "general solutions" in terms of a certain number of parameters but does not prove that these solutions are really general. It is pointed out by S. Chowla that the author has given a solution of the system

$$\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i, \quad \sum_{i=1}^{3} x_i^5 = \sum_{i=1}^{3} y_i^5,$$

namely, 49, 75, 107; 39, 92, 100. This result is new [cf. Hardy and Wright, Introduction to the Theory of Numbers, p. 333]. There are a few misprints in the writing of the exponents. *I. A. Barnett* (Cincinnati, Ohio).

Stern, Erich. Au sujet des produits magiques et panmagiques. Revista Mat. Timişoara 19, 137–140, 157–160 (1940). [MF 1300]

Fitting, F. Carrés panmagiques de n^4 cases pour toute valeur de n impair. Sphinx 9, 116–118 (1939). [MF 996]

Venkatasubbiah, G. Methods of constructing magic squares. Math. Student 7, 101–107 (1939). [MF 1444]

The author generalizes a known method of constructing bordered magic squares of odd order, and gives a construction for magic squares of order $4n$. *R. J. Walker.*

Rosser, Barkley and Walker, R. J. The algebraic theory of diabolic magic squares. Duke Math. J. 5, 705–728 (1939). [MF 813]

A diabolic square (d.s.) of order n is defined as a square matrix in which the n elements of each row, column and diagonal have the same sum. There are n rows, n columns, and $2n$ diagonals (since the arrangement of rows and columns is considered to be cyclic, as if the square were inscribed on a torus). It is shown that, if $n\geqq 4$, the permutations of the n^2 elements which preserve these $4n$ sets of n elements, apart from rearrangement, form a group whose order is $4n^2\varphi(n)$ or $8n^2\varphi(n)$ according as n is even or odd, $\varphi(n)$ being the number of integers less than n and prime to n. A primitive square is defined as a square matrix (A_{ij}) in which $A_{ij}+A_{kl}=A_{il}+A_{kj}$. A regular square is defined as any transform of a primitive square, that is, as a matrix (B_{ij}), where $B_{ai+cj,bi+dj}=A_{ij}$ and (A_{ij}) is primitive. It is shown that (B_{ij}) is a d.s. whenever $abcd(a^2-b^2)(c^2-d^2)$ is prime to n, and that for $n=4$ or 5 there is no other kind of d.s. A numerical diabolic square (n.d.s.) is defined as a

d.s. whose elements are the numbers $1, 2, \cdots, n^2$. It is shown that there are exactly $(n!)^2(n-3)(n-4)$ regular n.d.s. whenever n is an odd prime. In particular, since every d.s. of order 5 is regular, there are 28800 n.d.s. of order 5. Regular n.d.s. occur whenever n is a multiple of 4, or odd but greater than 3, and irregular n.d.s. occur for all such values except 4 and 5. *H. S. M. Coxeter* (Toronto, Ont.).

Thébault, V. Sur des nombres curieux. Supplement to Mathesis **53**, nos. 7–8, 16 pp. (1939). [MF 856]

This paper is concerned chiefly with the digital properties of certain squares. Sequences of curious squares like $69^2=4761$, $669^2=447561$, $6669^2=44475561$, $\cdots$ are discussed in systems of numeration in which the base is of the form m^2+1. There is also a three page list of all sets of four numbers like 32,895,40,716, made up of all the digits from 0 to 9 and such that $32\cdot 895=40\cdot 716$. *D. H. Lehmer.*

Vijayaraghavan, T. On the irrationality of a certain decimal. Proc. Indian Acad. Sci., Sect. A. **10**, 341 (1939). [MF 908]

The decimal $.2357111317\cdots$, where the sequence of digits is that of the primes p in ascending order, is irrational. For if this were a recurring decimal with k digits in the recurring part, and l digits in the nonrecurring part, then for any s the number of primes with s digits would not exceed $K=sk+l$ (the author has $k+l$), and $\sum(1/p)\leqq\sum_s K/10^{s-1}<\infty$. Compare proofs in Hardy and Wright, The Theory of Numbers, p. 112. *G. Pall* (Montreal, Que.).

Nakayama, Masayosi. On the decomposition of a rational number into "Stammbrüche." Tôhoku Math. J. **46**, 1–21 (1939). [MF 1162]

A positive proper fraction a/b can be expressed as the sum of reciprocals of natural numbers, usually in several ways. This paper deals with the minimum number $N(a, b)$ of reciprocals necessary for a given fraction. For example, necessary conditions that $N(a, b)=a$ are derived. The number of fractions $G(a, n, x)$ with $a<b\leqq x$ such that $N(a, b)>n$ $(2\leqq n<a)$ is also estimated. *H. W. Brinkmann.*

Korselt, A. Ein Beweis für den Fundamentalsatz der Zahlentheorie. Jber. Deutsch. Math. Verein. Abt. 2. **49**, 73–74 (1940). [MF 1220]

After referring to Zermelo's remarkably simple proof of unique factorization in $R(1)$, the author proves by induction the theorem: For p a prime, and $p|ab$, we have $p|a$ or $p|b$ or both. Let p be the smallest prime for which $ab=cp$ and p does not divide either a or b. Then $a\equiv a'$, $b\equiv b'$ (mod p), $0<a'<p$, $0<b'<p$, $a'b'=c'p$, $c'<p$. Divide the last equation successively by all prime divisors of c', which by assumption are divisors of a' or of b', obtaining $a''b''=p$, $a'', b''<p$. *A. J. Kempner* (Boulder, Colo.).

Barnett, I. A. and Szász, Otto. On a certain Diophantine problem. Amer. Math. Monthly **46**, 545–554 (1939). [MF 675]

The authors determine all real solutions x, y of the equation $\cos nx+\cos ny=0$, for which $\cos x$ and $\cos y$ are rational, and all integers n for which such solutions exist. In generalization, the authors treat the case of certain systems of similar equations. Generalizing in another direction, the authors deal, for an arbitrary integer s, with the equation $\cos snx+\cos ny=0$. *E. Rothe* (Oskaloosa, Iowa).

Rédei, L. Die Diophantische Gleichung $mx^2+ny^2=z^4$. Monatsh. Math. Phys. **48**, 43–60 (1939). [MF 625]

Let m and n be quadratfrei integers, $mn\neq-1$; assume that the greatest common divisor d of m and n is 1 or 2, and assume that in the latter case $mn\equiv 4$ (mod 16). The author studies the solvability of the Diophantine equation

$$mx^2+ny^2=z^4, \tag{1}$$

where z is odd and where x, y are relatively prime integers in the case $d=1$, and where x, y are rational numbers in the case $d=2$ such that $2x$ and $2y$ are relatively prime integers. Let D be the discriminant of a quadratic field. The author calls $D=D_1D_2$ a D-decomposition of the second kind [D-Zerfällung zweiter Art], if D_1 and D_2 are also discriminants, and if for all primes p of D either Kronecker's symbol $(D_1/p)=1$ or $(D_2/p)=1$. Let a_1, a_2 be discriminants of quadratic fields, $a_3>1$ quadratfrei, and $k_1=P(a_1^{\frac{1}{2}})$. Suppose that it is possible to choose an integer α_2 in k_1 and an ideal $\mathfrak{a}_3$ in k_1 as follows: (1) The norm of α_2 is $N_{k_1}(\alpha_2)=a_2g^2$, where g is a rational integer. (2) If we put $k_{12}=k_1(\alpha_2^{\frac{1}{2}})$, then the norm of the relative discriminant of k_{12} over k_1 is $N_{k_1}(D_{k_{12}|k_1})=a_2$. (3) If $a_3=p_1p_2\cdots p_r$ is the factorization of a_3, let $\mathfrak{a}_3$ be the product of the ideals $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_r$, where $\mathfrak{p}_\rho$ is a prime ideal and a divisor of p_ρ $(\rho=1, 2, \cdots, r)$. (4) Suppose that $(\alpha_2/\mathfrak{a}_3)$ has a value independent of the special choice of α_2 and $\mathfrak{a}_3$, and also that it is not zero. Then the author defines

$$\{a_1, a_2, a_3\}=(\alpha_2/\mathfrak{a}_3)=\pm 1.$$

The conditions for the existence of this symbol and methods for its determination were given by the author previously [J. Reine Angew. Math. **180**, 1–43 (1939)].

Using these definitions the author proves: Equation (1) has solutions in the sense defined above if and only if $\{D_1, D_2, m\}=+1$ for every D-decomposition $D=D_1D_2$ of the second kind admitted by the discriminant D of the quadratic field $P((-mn)^{\frac{1}{2}})$. This result shows that the solvability of (1) essentially depends on the solvability of certain ternary homogeneous quadratic Diophantine equations. Moreover the author considers the equation $mx^2+ny^2=z^l$. *A. Brauer* (Princeton, N. J.).

Townes, S. B. Table of reduced positive quaternary quadratic forms. Ann. of Math. **41**, 57–58 (1940). [MF 1004]

Employing conditions for a unique representative of each class of equivalent positive quaternary quadratic forms obtained in his dissertation [Chicago, 1936, unpublished. Cf. also B. W. Jones, Ann. of Math. **40**, 92–119 (1939)] by carrying through the Eisenstein method of reduction, the author computes a table of such representative forms for determinants 1 to 25. One would have liked to see included some indication of the method of construction used. The thesis indicates [p. 65] that the author is in possession of a table extending up to determinant 50. *A. E. Ross* (St. Louis, Mo.).

Taussky, Olga and Todd, John. A characterisation of algebraic numbers. Proc. Roy. Irish Acad., Sect. A. **46**, 1–8 (1940). [MF 1296]

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be a basis of an algebraic number field K and β any number in K; we can then express $\beta\alpha_i$ in the form $\beta\alpha_i=a_{i1}\alpha_1+a_{i2}\alpha_2+\cdots+a_{in}\alpha_n$. If we associate the resulting matrix $A=(a_{ij})$ with β, we obtain a matrix representation of K, known as the regular representation of K with respect to the given basis. Now let a matrix A of rational numbers be given; the question is, does there exist

a field K with such a basis that, for some β in K, A is associated with β. The authors prove that this is the case if and only if either the characteristic polynomial of A is irreducible or is equivalent to the direct sum of several matrices each of which has the same irreducible characteristic polynomial $f(\lambda)$; in the second case the characteristic polynomial of A is of course a power of $f(\lambda)$. It is also shown that the characteristic polynomial of the "general" element in a regular matrix representation of a hypercomplex system over the rational field is always reducible if the system is not isomorphic to an algebraic number field.

H. W. Brinkmann (Swarthmore, Pa.).

Bauer, Mihály. Über zusammengesetzte relativ Galois'sche Zahlkörper. Mat. Fiz. Lapok **46**, 127–133 (1939). (Hungarian) [MF 983]

Let $G_1(k)$ and $G_2(k)$ be relative Galois fields over k. Let $\mathfrak{p}|p$, where $\mathfrak{p}$ is a prime ideal of k. Let the degree of $\mathfrak{p}$ in $G_1(k)$, $G_2(k)$, $G_1G_2(k)$ be f_1, f_2, F, respectively. Let the order of $\mathfrak{p}$ in $G_1(k)$, $G_2(k)$, $G_1G_2(k)$ be g_1, g_2, G, respectively. Write $g_i=p^{m_i}g_i^{(0)}$; $(p, g_i^{(0)})=1$; $G=p^MG^{(0)}$; $(p, G^{(0)})=1$. Then the author proves that

$$G=\frac{g_1g_2}{\Delta},\quad F=\frac{f_1f_2}{(f_1,f_2)}\Delta',\quad G^{(0)}=\frac{g_1^{(0)}g_2^{(0)}}{(g_1^{(0)},g_2^{(0)})},$$

where Δ is a divisor of (g_1, g_2) and $\Delta'|\Delta$. A special case of these formulas was proved by Ore. *P. Erdös.*

Bauer, Mihály. Über die Zusammensetzung algebraischer Zahlkörper. Mat. Fiz. Lapok **46**, 134–140 (1939). (Hungarian) [MF 984]

Let $K_1(k)$ and $K_2(k)$ be algebraic number fields over k. Let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be prime ideals of $K_1(k)$, $K_2(k)$, respectively, and let $\mathfrak{P}$ be a prime ideal of $K_1K_2(k)$ with $\mathfrak{P}|\mathfrak{P}_1$, $\mathfrak{P}|\mathfrak{P}_2$. Let $\mathfrak{p}|p$, p rational prime. The degrees of the prime ideals $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}$ are f_1, f_2, F, respectively, and their orders are g_1, g_2, G. Then the author proves that

$$G=\frac{g_1g_2}{(g_1,g_2)}\cdot t,\quad t\leqq(g_1,g_2);\quad F=\frac{f_1f_2}{(f_1,f_2)}\cdot U,\quad U\leqq\frac{(g_1,g_2)}{t};$$

$$G^{(0)}=\frac{g_1^{(0)}g_2^{(0)}}{(g_1^{(0)},g_2^{(0)})}\cdot r,\quad r\leqq p^{m_1+m_2-M};\quad (r,p)=1,$$

where t, U, r are rational integers, and where $g_i^{(0)}$ $(i=1,2)$, $G^{(0)}$ are defined by $g_i=p^{m_i}g_i^{(0)}$; $G=p^MG^{(0)}$; $(g_i^{(0)}, p)=(G^{(0)}, p)=1$. These results are the generalizations of those of Mikao Moriya. *P. Erdös* (Princeton, N. J.).

Scott, S. A. Elementary methods in the theory of numbers. Edinburgh Math. Notes **1939**, no. 31, xvi–xxiii (1939). [MF 993]

The author obtains by neat elementary methods the inequality

$$\frac{c_2x}{\log x}<\pi(x)<\frac{c_3x}{\log x}$$

for the prime number junction, and several allied inequalities. The reader is assumed to know only the most elementary properties of $\log x$, a^x, and the fact that a bounded monotonic sequence approaches a limit. *M. Ward.*

Pillai, S. S. On the smallest prime of the form $km+l$. Proc. Indian Acad. Sci., Sect. A. **10**, 388–389 (1939). [MF 910]

Some simple remarks about the smallest primes of arithmetical progressions. The formulations are not always correct. *P. Scherk* (Watertown, Conn.).

Turán, Paul. Über die Primzahlen der arithmetischen Progression. (II.) Acta Litt. Sci. Szeged **9**, 187–192 (1939). [MF 610]

Let $P(k,l)$ denote the smallest prime of the form $kx+l$. It was proved by Chowla [J. Indian Math. Soc. (2) **1**, 1–3 (1934)] that if no L-function vanishes for $\sigma>1/2$, then $P(k,l)<c_1\phi(k)^{2+\epsilon}$, where ϵ is arbitrarily small and $c_1=c_1(\epsilon)$. The author now shows that a similar result follows from a weaker hypothesis. Suppose that there are constants δ, $0<\delta\leqq1/2$, and α, such that no L-function vanishes for $1-\delta<\sigma\leqq2$, $|t|\leqq\alpha$. Then $P(k,l)<c_3\phi(k)^{c_4}$, where c_3 is an absolute constant, and $c_4=c_4(\alpha,\delta)$, $c_4>1/\delta$.

E. C. Titchmarsh (Oxford).

Sambasiva Rao, K. On a particular representation of integers as sums of kth powers. J. Indian Math. Soc. **3**, 262–265 (1939). [MF 1022]

The author considers the unique representation of a positive integer x in the form $x=x_1^k+x_2^k+\cdots+x_s^k$, where x_i^k is the largest kth power not exceeding $x-x_1^k-\cdots-x_{i-1}^k$, $1\leqq i\leqq s-1$, and $x_s^k=x-x_1^k-\cdots-x_{s-1}^k$. Two inequalities are established: (1) $s<c_1\log\log x$ for all $x>x_0$; (2) $s>c_2\log\log x$ for infinitely many values of x. These results are obtained by manipulating the inequalities involving x_i, $1\leqq i\leqq s$, but the author's proof of (1) is incorrect.

R. D. James (Saskatoon, Sask.).

Pillai, S. S. A note on the paper of Sambasiva Rao. J. Indian Math. Soc. **3**, 266–267 (1939). [MF 1023]

In the notation of the preceding paper by Sambasiva Rao, the following result is obtained: $\max s=c(k)\log\log x+M(k)+O(k\log k)$, where $c(k)=\{\log k-\log(k-1)\}^{-1}$ and $M(k)=\max s$ for $1\leqq x\leqq(2k)^k$. The method of proof is similar to that employed by Rao but more precise approximations are used. *R. D. James* (Saskatoon, Sask.).

Pillai, S. S. On the number of representations of a number as the sum of the square of a prime and a squarefree integer. Proc. Indian Acad. Sci., Sect. A. **10**, 390–391 (1939). [MF 911]

Let $R(n)$ denote the number of representations of n as the sum of the square of a prime and a squarefree integer; $n\not\equiv1 \pmod 4$. Following Erdös's proof of $R(n)>\sigma$ [J. London Math. Soc. **10**, 243–245 (1935)], the author proves

$$R(n)=\frac{2\sqrt{n}}{\lg n}\prod_q\left(i-\frac{2}{q(q-1)}\right)+O\left(\frac{\sqrt{n}}{\lg n\lg\lg n}\right),$$

where q runs through all the primes for which n is a quadratic residue. The paper contains inaccuracies.

P. Scherk (Waterville, Conn.).

Pillai, S. S. On numbers which are not multiples of any other in the set. Proc. Indian Acad. Sci., Sect. A. **10**, 392–394 (1939). [MF 912]

Given a set of positive integers b, none of them divisible by any other of the set. F. Behrend proved

$$\sum_{b\leqq x}\frac{1}{b}=O\left(\frac{\lg x}{(\lg\lg x)^{\frac{1}{2}}}\right)$$

[J. London Math. Soc. **10**, 42–44 (1935)]. By choosing as b's the products of n (equal or different) prime factors, and

putting $n=[\lg \lg x]$, the author proves: There exists a constant c and to each $x \geqq 3$ a set of b's, such that

$$\sum_{b \leqq x} \frac{1}{b} \geqq \frac{c \cdot \lg x}{(\lg \lg x)^{\frac{1}{2}}}.$$

More exactly: To every $\delta>0$, there is an x_0 such that

$$\sum_{b \leqq x} \frac{1}{b} \geqq \left(\frac{c^{B}}{4(2\pi)^{\frac{1}{2}}}+O(\delta)\right) \frac{\lg x}{(\lg \lg x)^{\frac{1}{2}}},$$

where

$$B=\lim_{z \to \infty}\left(\sum_{p \leqq z} \frac{1}{p}-\lg \lg z\right), \qquad p \text{ prime},$$

for every $x \geqq x_0$ and the mentioned set of b's.

P. Scherk (Watertown, Conn.).

Kac, M. Almost periodicity and the representation of integers as sums of squares. Amer. J. Math. **62**, 122–126 (1940). [MF 963]

The author discusses almost periodic properties of the sequence $f_s(n)=n^{1-\frac{1}{2}s}r_s(n)$, where $r_s(n)$ is the number of representations of n as a sum of s squares, in relation to the Hardy-Littlewood singular series

$$\rho_s(n)=\frac{\Pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \sum_{h,k;\ 0 \leqq h<k;\ (h,k)=1} k^{-s} S_{hk}^{s}\, e\left(-\frac{hn}{k}\right),$$

where

$$S_{hk}=\sum_{j=0}^{k-1} e\left(\frac{hj^2}{k}\right), \quad e(x)=e^{2\pi i x},$$

regarded as a formal Fourier series $\sum_\lambda a_s(\lambda)e(-\lambda n)$ with

$$a_s(\lambda)=\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} f_s(n)e(\lambda n), \qquad 0 \leqq \lambda<1.$$

For $5 \leqq s \leqq 8$ it is inferred from Hardy's identity $f_s(n)=\rho_s(n)$ and the absolute convergence of $\rho_s(n)$ that $f_s(n)$ is a uniformly a.p. sequence. For $s=4$ it is proved (from Jacobi's formula for $r_4(n)$), and for $s=3$ it is stated, that $f_s(n)$ is a.p. (B^2). It is suggested that the same should be true for $s>8$. [This is so since $f_s(n)=\rho_s(n)+o(1)$ as $n \to \infty$.] It is shown that $f_2(n)=r_2(n)$ is not even a.p. (B^1).

A. E. Ingham (Berkeley, Calif.).

Rademacher, Hans. Fourier expansions of modular forms and problems of partition. Bull. Amer. Math. Soc. **46**, 59–73 (1940). [MF 1247]

This paper consists of the address delivered by the author before the American Mathematical Society, December, 1938. It is an expository paper and, aside from a brief history of the problem of partitions, it is concerned mainly with the work that has recently been done on modular forms of positive dimension and the resulting formulas for partition functions. The theory of modular forms of positive dimension has not been as well developed as that for negative dimension. Some points in which this theory is not yet complete are mentioned as well as some other related unsolved problems. A large part of the recent developments were initiated by the author's solution of the problem of unrestricted partitions and many of the results are due either to him or to his direct inspiration. Numerous references to the relevant literature are given.

H. S. Zuckerman (Seattle, Wash.).

Gut, Max. Folgen von Dedekindschen Zetafunktionen. Monatsh. Math. Phys. **48**, 153–160 (1939). [MF 635]

Let k be an algebraic field of infinite degree defined by the sequence $k_1 \subset k_2 \subset \cdots$, $\lim_{i\to\infty} k_i=k$, where k_i is a field of finite degree n_i, such that each of the prime ideals of k_i, which divides one of the first i primes $2, 3, 5, \cdots, p_i$ can be divided in k_{i+1} into prime ideals of relative order and relative degree one [see H. Hasse, Math. Ann. **95** (1926), theorem 1]. The author proves: (1) If $\zeta_i(s)$ is the Dedekind zeta function of k_i, then $\lim_{i\to\infty}(\zeta_i(s))^{1/n_i}=Z(s)$, where $Z(s)$ is analytic and not zero for $\sigma>1$ ($s=\sigma+it$). (2) If k is formed by adjoining to the field of rationals, in any order, all absolute cyclic fields of odd prime degree q, then for $\sigma>1/q$

$$Z(s)=\Big\{\prod_{\substack{p=q\\ p\equiv 1 \pmod q}} (1-p^{-qs})^{q-1}\cdot\zeta^{q}(qs)\Big\}^{q^{-2}}.$$

(3) If k is formed by adjoining to the field of rationals all absolute quadratic fields, then for $\sigma>\frac{1}{2}$,

$$Z(s)=\{\zeta^2(2s)(2^{2s}-1)2^{-2s}\}^{1/2}.$$

H. S. A. Potter (Aberdeen).

Jarnik, Vojtěch. Über einen p-adischen Übertragungssatz. Monatsh. Math. Phys. **48**, 277–287 (1939). [MF 645]

Let $\alpha_1, \cdots, \alpha_m$ be m p-adic integers, $\beta_1(\alpha_1, \cdots, \alpha_m)$ and $\beta_2(\alpha_1, \cdots, \alpha_m)$ the upper bounds of all exponents γ and δ such that the problems

$$(x_1, \cdots, x_m, p)=1, \quad \max(|x_0|, \cdots, |x_m|) \leqq \xi,$$
$$|x_0+\alpha_1 x_1+\cdots+\alpha_m x_m|_p \leqq \xi^{-\gamma},$$

or

$$(y_0, p)=1, \quad \max(|y_0|, \cdots, |y_m|) \leqq \eta,$$
$$|\alpha_i y_0-y_i|_p \leqq \eta^{-\delta}, \qquad i=1, 2, \cdots, m,$$

have integral solutions x_i or y_i for an infinity of arbitrarily large ξ or η. It is easily proved that $\beta_1 \geqq m+1$, $\beta_2 \geqq (m+1)/m$, and that $\beta_1=\beta_2$ for $m=1$. K. Mahler has shown [Časopis Pěst. Mat. Fys. **68**, 85–92 (1939)] that

$$\beta_2 \geqq \frac{m\beta_1}{1+(m-1)\beta_1}.$$

Author shows now that this is the best possible result: To every $\gamma \geqq m+1$ there exist m p-adic integers $\alpha_1, \cdots, \alpha_m$ such that

$$\beta_1=\gamma, \quad \beta_2=\frac{m\gamma}{1+(m-1)\gamma}.$$

The proof depends on the following lemmas: (1) To a given $\gamma \geqq 2$ there is a p-adic integer α for which $\beta_1(\alpha)=\gamma$. (2) Let $\beta_1(\alpha_1)=\gamma \geqq 2$. Then for all p-adic integers $\alpha_2, \cdots, \alpha_m$

$$\beta_1(\alpha_1, \cdots, \alpha_m) \geqq \max(m+1, \gamma),$$

$$\beta_2(\alpha_1, \cdots, \alpha_m) \geqq \max\left(\frac{m+1}{m}, \frac{m\gamma}{1+(m-1)\gamma}\right),$$

and for "nearly all" these integers

$$\beta_1(\alpha_1, \cdots, \alpha_m)=\max(m+1, \gamma),$$

$$\beta_2(\alpha_1, \cdots, \alpha_m)=\max\left(\frac{m+1}{m}, \frac{m\gamma}{1+(m-1)\gamma}\right).$$

For the method, compare author's paper [Prace Mat.-Fiz. **43**, 151–166 (1936)]; there the analogous problem in the real case, that is, for Khintchine's "Uebertragungssatz," is solved, with identical result.

K. Mahler (Manchester).

Koksma, J. F. Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen. Monatsh. Math. Phys. **48**, 176–189 (1939). [MF 638]

Let $M(n, a)$, for integral $n\geqq 1$, $a\geqq 1$, be the set of all polynomials

$$f(x)=\sum_{k=0}^{n} a_k x^k \not\equiv 0$$

of degree not greater than n with integral coefficients of absolute value not greater than a. Further, let θ be a real or complex transcendental number, $\omega(n, a)$ the minimum of $|f(\theta)|$ taken over all elements of $M(n, a)$, and $\omega^*(n, a)$ the minimum of $|\theta-\alpha|$, where α is the root of any one of the primitive irreducible polynomials belonging to $M(n, a)$. Put

$$\omega(n)=\limsup_{a\to\infty}\frac{\log(1/\omega(n, a))}{\log a},$$

$$\omega^*(n)=\limsup_{a\to\infty}\frac{\log(1/(a\omega^*(n, a)))}{\log a},$$

and

$$\omega=\limsup_{n\to\infty}\frac{\omega(n)}{n},\qquad \omega_1=\operatorname*{upper\ bound}_{n\geqq 1}\frac{\omega(n)}{n};$$

$$\omega^*=\limsup_{n\to\infty}\frac{\omega^*(n)}{n},\qquad \omega_1^*=\operatorname*{upper\ bound}_{n\geqq 1}\frac{\omega^*(n)}{n},$$

and denote by μ (μ^*) the number ∞, if $\omega(n)$ ($\omega^*(n)$) is finite for all n, otherwise the smallest index for which this function is infinite. K. Mahler [J. Reine Angew. Math. **166**, 118–136 (1932)] called θ an S-, T- or U-number, according as $\omega<\infty$, $\mu=\infty$, $\omega=\mu=\infty$, or $\omega=\infty$, $\mu<\infty$; the author analogously defines S^*-, T^*-, and U^*-numbers by means of ω^* and μ^* instead of ω and μ. Then he shows that θ has the same letter in both classifications by proving that

$$\mu=\mu^*\quad\text{and}\quad \omega_1^*\leqq\omega_1\leqq\omega_1^*+2$$

(the numbers ω and ω_1 (ω^* and ω_1^*) are simultaneously finite or infinite). He further shows that for "nearly all" real (complex) numbers $\omega_1^*\leqq 1$ ($\omega_1^*\leqq 1/2$), and therefore $\omega_1\leqq 3$ ($\omega_1\leqq 5/2$); this forms a slight improvement on a result of Mahler [Math. Ann. **106**, 131–139 (1932)]. The proof of the surprising inequality $\omega_1\leqq\omega_1^*+2$ depends on the lemma: "Let r be any integer in $(1, n)$, $m^{(1)}, \cdots, m^{(r)}$ any r positive integers of sum not greater than n, and $a^{(1)}, \cdots, a^{(r)}$ any r positive integers of product not greater than $(4n)^n a$. If $\Omega(n, a)$ is the minimum of all expressions

$$\prod_{\rho=1}^{r}\omega^*(m^{(\rho)}, a^{(\rho)}),$$

then

$$\Omega(n, a)\leqq Ba^{2n-3}\omega(n, a),$$

where B is a positive constant depending only on θ and n." It would be of interest to replace the exponent $2n-3$ of a by a smaller number. *K. Mahler* (Manchester).

ANALYSIS

Mandelbrojt, Szolem. Sur les fonctions convexes. C. R. Acad. Sci. Paris **209**, 977–978 (1939). [MF 1197]

A single-valued function $f(x)$ ($-\infty<x<\infty$; $-\infty<f(x)\leqq\infty$; $f(x)\not\equiv\infty$) is called convex if $x_1<x_2<x_3$ implies $(x_3-x_1)f(x_2)\leqq(x_3-x_2)f(x_1)+(x_2-x_1)f(x_3)$. The following theorem is proved. If $\phi(t)$ is a single-valued function ($-\infty<t<\infty$; $-\infty<\phi(t)<\infty$; $\phi(t)\not\equiv\infty$), then

$$f(x)=\operatorname*{l.u.b.}_{-\infty<t<\infty}(xt-\phi(t)) \tag{1}$$

is convex. Conversely, any given convex function $f(x)$ can be put in the form (1) with some appropriate $\phi(t)$, a particular (convex) $\phi(t)$ satisfying (1) being

$$\phi(t)=\operatorname*{l.u.b.}_{-\infty<x<\infty}(xt-f(x)).$$

This is a reciprocal relationship associating convex functions in pairs. *I. J. Schoenberg* (Waterville, Me.).

Szegö, G. On the gradient of solid harmonic polynomials. Trans. Amer. Math. Soc. **47**, 51–65 (1940). [MF 1069]

In a previous paper [Schr. Königsberg. Gel. Ges. **1928**, 59–70] the author proved the following theorem: Let $u(x, y)$ be a real harmonic polynomial of the nth degree which satisfies the inequality $|u(x, y)|\leqq 1$ in $x^2+y^2\leqq 1$. Then $|\text{grad } u|=(u_x^2+u_y^2)^{\frac{1}{2}}\leqq n$, $x^2+y^2\leqq 1$. In the present paper the author extends his result to the three-dimensional case and proves that if $u(x, y, z)$ is a real harmonic polynomial of the nth degree which satisfies the inequality $|u(x, y, z)|\leqq 1$ in the unit sphere $x^2+y^2+z^2\leqq 1$, then $|\text{grad } u|\leqq\rho_n$, $n\geqq 4$, where

$$\rho_n=2n\left(1-\tfrac{1}{3}+\cdots+\frac{(-1)^{n-1}}{2n-1}\right)-\{1-(-1)^n\}/2.$$

The equality sign holds only for polynomials whose boundary values for $x^2+y^2+z^2=1$ are of the form $\pm\cos n\gamma$, where γ is the spherical distance of the variable point (x, y, z) on the unit sphere from a fixed point P_0. For these polynomials the equality sign holds only at P_0 and at the point diametrally opposite to P_0. In the case $n=2, 3$ a less precise result is obtained, $|\text{grad } u|\leqq 2^{\frac{1}{2}}n$. The method of the proof is analogous to that used in the two-dimensional case. The main difficulty consists in proving the positiveness for all θ of the polynomial

$$M_n(\theta)=1+2\sum_{\nu=1}^{n-1}\{(\rho_{n-\nu}/\rho_n)\cos n\theta\cos(n-\nu)\theta$$
$$+((n-\nu)/n)\sin n\theta\sin(n-\nu)\theta\}.$$

J. D. Tamarkin (Providence, R. I.).

Gorny, A. Contribution à l'étude des fonctions dérivables d'une variable réelle. Acta Math. **71**, 317–358 (1939). [MF 768]

The author proves certain inequalities previously announced in shorter papers, but in modified form, as suggested by work of other authors subsequent to his first papers. His first result states that, if $f(x)$ is n times differentiable on a closed interval I of length δ, and if $|f(x)|\leqq M_0$, $|f^{(n)}(x)|\leqq M_n$, then, in I and for $0<k<n$,

$$|f^{(k)}(x)|<4e^{2k}\binom{n}{k}^k M_0^{1-(k/n)}M_n'^{k/n},$$

while, at the midpoint of I,

$$|f^{(k)}(x)|<16(2e)^k M_0^{1-(k/n)}M_n'^{k/n},$$

where

$$M_n'=\max(M_n, M_0 n!\delta^{-n}).$$

If I is the interval $(0, \infty)$ or $(-\infty, \infty)$, then clearly $M_n'=M_n$ and for the interval $(-\infty, \infty)$ there is the further refine-

ment that any point can be considered the midpoint. This theorem allows the author to obtain necessary and sufficient conditions that two classes of functions (each class depending on the infinite sequence of upper bounds for successive derivatives of functions in the class) are equivalent on the interval $(-\infty, \infty)$. He also proves that the product of two functions in a class belongs to the same class.

The author proves an unrelated inequality for functions represented by a Stieltjes integral which is the analogue of the theorem in Fourier series that, if $f(x) \sim \sum a_k e^{ikx}$ and $|f^{(n)}(x)| \leqq M_n$, then $a_k = O(M_n/k^n)$ as $|k| \to \infty$. The author then proves several theorems which are a generalization to sequences of analytic functions of certain theorems of S. Bernstein and de la Vallée Poussin on the differentiability of limit functions of sequences of polynomial and trigonometric functions. *N. Levinson* (Cambridge, Mass.).

Fourier Series and Integrals, Theory of Approximation

Salem, Raphaël. Sur les propriétés descriptives des ensembles de points de divergence des séries trigonométriques. C. R. Acad. Sci. Paris **209**, 748–750 (1939). [MF 843]

If $r_n \geqq 0$ and $\sum_1^\infty r_n$ diverges, then a sequence $s_n \geqq 0$ can be found such that the trigonometrical series

$$\sum_1^\infty r_n e^{2\pi i n x - s_n}$$

diverges in a set of the second category. Compare C. R. Acad. Sci. Paris **205**, 832 (1937). *W. W. Rogosinski.*

Szász, Otto. On the order of the partial sums of Fourier power series. Bull. Amer. Math. Soc. **46**, 108–112 (1940). [MF 1255]

Let $f(x)$ be periodic (of period 2π) and integrable. It is well known that the partial sums $s_n(x)$ of the Fourier series expansion of $f(x)$ are $o(n)$ and that this estimate cannot be improved for the class of integrable functions. In the present note the author shows, by using two special sequences of polynomials, that this estimate cannot be improved even for the class of integrable power series, that is, for the class of functions $f(x)$ for which $f(x) \sim \sum_{n=0}^\infty c_n e^{inx}$. Using the same method the author constructs power series which are generalized Fourier series (the coefficients are expressed by means of improper integrals), the coefficients of which do not admit of a better estimate than $o(n)$. Analogous result for Fourier series was given by Titchmarsh by an entirely different method [Proc. London Math. Soc. (2) **22**, 25–26 (1924)]. *J. D. Tamarkin* (Providence, R. I.).

Szász, Otto. On the Cesàro and Riesz means of Fourier series. Compositio Math. **7**, 112–122 (1939). [MF 371]

The author considers Cesàro transforms

$$C_n^{(k)} = \binom{n+k}{k}^{-1}\left\{\binom{n+k}{k}u_0 + \cdots + \binom{k}{k}u_n\right\}$$

and Riesz transforms

$$R_n^{(k)} = (n+1)^{-k}\{(n+1)^k u_0 + n^k u_1 + \cdots + u_n\}$$

of a series $\sum_0^\infty u_\nu$. He establishes fundamental relations

$$R_n^{(2)} = (n+1)^{-1}\{nC_{n-1}^{(2)} + C_n^{(1)}\}$$
$$= (2n+2)^{-1}\{nC_{n-1}^{(2)} + (n+r)C_n^{(2)}\}$$
$$C_{2m}^{(3)} = \binom{2m+3}{3}^{-1} \sum_{\nu=0}^m (2\nu+1)^2 R_{2\nu}^{(2)},$$
$$C_{2m-1}^{(3)} = \binom{2m+2}{3}^{-1} \sum_{\nu=1}^m (2\nu)^2 R_{2\nu-1}^{(2)}.$$

It is proved that the $(R, 2)$ transforms of the series $\sum_0^\infty \sin na \sin nx$ are positive for $0<a<\pi$, $0<x<\pi$, which leads to the following results. (i) If $0 \leqq f(x) \leqq M$, $0<x<\pi$, then the $(R, 2)$ transforms of the sine series of $f(x)$ satisfy the inequality $0 \leqq R_n^{(2)}(x) \leqq B_n(x)$, where $B_n(x)$ are $(R, 2)$ transforms of the sine series expansion of the constant M. (ii) If $f(x)>0$ in $0<x<\pi$ and is convex upwards, so are the $(R, 2)$ transforms of the sine series of $f(x)$. (iii) If $f(x)$ is monotone in $0<x<\pi$, so are the $(R, 2)$ transforms of the cosine series of $f(x)$. Analogous results were established by Fejér [Z. Angew. Math. Mech. **13**, 80–88 (1933); J. London Math. Soc. **8**, 53–62 (1933); J. Math. Phys. Mass. Inst. Tech. **13**, 1–17 (1934)] for the $(C, 3)$ transforms. They can be derived from the above results, but do not hold for the $(C, 2)$ transforms. It is proved, however, that if $\sum_\nu u_\nu$ is $(R, 2)$ summable then a necessary and sufficient condition for it to be $(C, 2)$ summable is that

$$\liminf_{n\to\infty} n^{-3} \sum_{\nu=0}^n (n-\nu+1)\nu u_\nu \geqq 0.$$

Since this condition is satisfied for $u_n \to 0$, it follows that the methods $(R, 2)$ and $(C, 2)$ are equivalent for Fourier series (although they are not equivalent in general). *J. D. Tamarkin* (Providence, R. I.).

Izumi, Shin-ichi and Kawata, Tatsuo. Notes on Fourier series (IX): Uniform Cesàro summability. Tôhoku Math. J. **46**, 117–122 (1939). [MF 1174]

Let $-1<\alpha<1$ and $f(x)$ be continuous and periodic (of period 2π). By a direct investigation of the (C, α) transform of the Fourier series of f the author proves that the difference between $f(x)$ and this transform does not exceed $C[o(1)+T_n(x+\theta)+U_n(x-\theta)]$, where C is a constant, $o(1)$ tends uniformly to zero as $n\to\infty$, and, assuming n odd,

$$T_n(x) = \sum_{k=0}^{(n-1)/2} |f(x+2k\pi/n) - f(x+(2k+1)\pi/n)|(k+1)^{-1-\alpha},$$

$$U_n(x) = \sum_{k=0}^{(n-1)/2} |f(x-2k\pi/n) - f(x-(2k+1)\pi/n)|(k+1)^{-1-\alpha},$$

and $\pi/2n \leqq \theta \leqq \pi/2n+\pi/n$ in the case of $T_n(x+\theta)$ and $\pi/n \leqq \theta \leqq 2\pi/n$ in the case of $U_n(x-\theta)$. From this formula follow, as special cases, various criteria for uniform (C, α) summability of the Fourier series of $f(x)$ obtained previously by other authors. *J. D. Tamarkin* (Providence, R. I.).

Maruyama, Gisirô. Determination of the jump of a function by its Fourier series. Tôhoku Math. J. **46**, 68–74 (1939). [MF 1168]

An extension of the following theorem due to the reviewer: Let $\bar{\sigma}_n(x)$ be the $(C, 1)$ means corresponding to the conjugate of the Fourier series of a function $f(x)$; furthermore let

$$\lim_{h\to 0} h^{-1}\int_0^h \{f(x+t)-f(x-t)\}dt = D(x);$$

then
$$\lim_{n\to\infty}(\bar\sigma_{2n}(x)-\bar\sigma_n(x))=D(x)(\log 2/\pi).$$
The author replaces the pair $(2n, n)$ by (μ_n, λ_n), where $\mu_n>\lambda_n$ are integers and
$$\lim_{n\to\infty}(\mu_n/\lambda_n)=A, \qquad 1<A.$$
O. Szász (Cincinnati, Ohio).

Kawata, Tatsuo. A proof of a theorem of Hardy and Littlewood concerning strong summability of Fourier series. Proc. Imp. Acad., Tokyo **15**, 243–246 (1939). [MF 1141]

Let $f(x)\varepsilon L(-\pi, \pi)$, $\phi(t)=1/2[f(x+t)+f(x-t)-2f(x)]$. Hardy and Littlewood [Fund. Math. **25**, 162–189 (1935), Theorem 3] proved that if $\int_0^h|\phi(t)|dt=o(h)$, then
$$\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|^2=o(n\log n),$$
where $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$. In the present paper the author gives a simpler proof of this theorem, without using the theory of power series which was used by Hardy and Littlewood.
J. D. Tamarkin (Providence, R. I.).

Randels, W. C. On the absolute summability of Fourier series. II. Bull. Amer. Math. Soc. **46**, 86–88 (1940). [MF 1251]

A new example is constructed to prove that absolute summability $(C, 1)$ is not a local property of a Fourier series. *O. Szász* (Cincinnati, Ohio).

Hyslop, J. M. On the absolute summability of the successively derived series of a Fourier series and its allied series. Proc. London Math. Soc. **46**, 55–80 (1939). [MF 734]

Let $f(x)$ be integrable Lebesgue and periodic, of period 2π. Let there exist a polynomial $P(t)=\sum_{i=0}^{r-1}\theta_i t^i/i!$ such that the functions
$$g(t)=1/2t^{-r}\{[f(x+t)-P(t)]+(-1)^r[f(x-t)-P(-t)]\},$$
$$h(t)=1/2t^{-r}\{[f(x+t)-P(t)]-(-1)^r[f(x-t)-P(-t)]\}$$
are integrable in $(-\pi, \pi)$. Let
$$G_\alpha(t)=(1/\Gamma(\alpha))\textstyle\int_0^t(t-u)^{\alpha-1}g(u)du, \qquad \alpha>0,$$
$G_0(t)=g(t)$, $g_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}G_\alpha(t)$, $\alpha\geqq0$, with analogous notation for h instead of g. The main results of the paper are embodied in the following four theorems. (i) If $g_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the rth derived series of the Fourier series of $f(t)$ is absolutely summable $(C, \alpha+r+\delta)$ at $t=x$, $\alpha\geqq0$, $\delta>0$. (ii) If $H_\alpha(+0)=0$ and $\int_0^\pi u^{-\alpha}|dH_\alpha(u)|<\infty$, then the rth derived series of the conjugate of the Fourier series of $f(t)$ is absolutely summable $(C, \alpha+r+\delta)$ at $t=x$, $\alpha>0$, $\delta>0$. (iii) If the rth derived series of the Fourier series of $f(t)$ is absolutely summable (C, α) at $t=x$, then $g_{\alpha-r+1+\delta}(t)$ is of bounded variation in $(0, \pi)$, $\alpha\geqq r\geqq0$, $\delta>0$. (iv) If the rth derived series of the conjugate of the Fourier series of $f(t)$ is absolutely summable (C, α) at $t=x$, then $\int_0^\pi t^{-1}|h_{\alpha-r+\delta}(t)|dt<\infty$, $\alpha\geqq r\geqq0$, $\delta>0$. These theorems generalize results obtained by various authors in the case $r=0$ and for usual summability instead of absolute summability. The proofs are based on a considerable body of lemmas, some of them of independent interest, which cannot be reproduced here.
J. D. Tamarkin (Providence, R. I.).

Pitt, H. R. On Wiener's general harmonic analysis. Proc. London Math. Soc. **46**, 1–18 (1939). [MF 732]

The main purpose of the paper is to extend results of Wiener [Math. Z. **24**, 575–616 (1925); Acta Math. **55**, 117–258 (1930)] concerning representation of an "arbitrary" function $f(x)$ by formulas of the type
$$(*) \qquad f(x)\sim\int_{-\infty}^{\infty}e^{-ixs}ds(x),$$
and to investigate systematically relationships between (*) and classical developments as Fourier integrals and Fourier series of periodic and almost periodic functions. Let $\mathfrak{B}=\mathfrak{B}_p$ be the class of functions $f(x)$ integrable over every finite range and such that
$$\sum_{-\infty}^{\infty}F_n{}^p(|n|+1)^{-p}<\infty, \quad F_n=\int_n^{n+1}|f(x)|dx, \quad 1\leqq p\leqq2.$$
This class includes the class B_p of functions for which
$$(2T)^{-1}\int_{-T}^{T}|f(x)|^p dx, \qquad p>1,$$
is bounded for all T, as well as the class of periodic and almost periodic functions in the Stepanoff sense. It is shown that, if $f\varepsilon\mathfrak{B}$, then
$$s_l(y)=(2\pi)^{-1}\int_{-l}^{l}\frac{f(x)}{ix}(e^{ixy}-e^{x^2})dx$$
converges in mean of order p' over every finite range as $l\to\infty$. It follows that, if $s(y)$ is this limit and
$$S_y(x)=S(y)e^{-ixy}-s(-y)e^{ixy}+ix\int_{-y}^{y}e^{-ixt}s(t)dt,$$
$$\sigma_y(x)=\frac{1}{y}\int_0^y S_t(x)dt,$$
then
$$\sigma_y(x)=\frac{1}{2\pi y}\int_{-\infty}^{\infty}f(x-u)\left(\frac{\sin\frac{1}{2}yu}{\frac{1}{2}u}\right)^2du.$$
It is proved that if
$$f(x)=\int_{-\infty}^{\infty}e^{-ixs}ds^*(x),$$
where $s^*(x)$ is of bounded variation over $(-\infty, \infty)$, then the difference $s(x)-s^*(x)$ is constant almost everywhere, with analogous results in the cases where $f(x)\varepsilon L^p(-\infty, \infty)$, or where $f(x)$ is almost periodic. Among the author's results concerning the convergence and summability of the integral (*) it should be mentioned that if $f\varepsilon\mathfrak{B}$ then almost everywhere $\sigma_y(x)\to f(x)$ as $y\to\infty$. Finally the author establishes various inequalities which are extensions to the present general situation of the inequalities of M. Riesz and Young-Hausdorff, well-known in the theory of Fourier series. *J. D. Tamarkin* (Providence, R. I.).

Erdös, Paul. On the smoothness properties of a family of Bernoulli convolutions. Amer. J. Math. **62**, 180–186 (1940). [MF 967]

Continuing a former investigation [these Rev. **1**, 52 (1940)], where further references are given, the author proves that for every integer m there exists a positive $\delta(m)$ such that the set of points a of the interval $1<a<1+\delta(m)$, for which
$$L(u;\sigma_a)=\Pi\cos(u/a^n)=o(|u|^{-m})$$
does not hold, is of measure zero. In other words, for almost

every a in the interval $(1, 1+\delta(m))$ the distribution function $\sigma_a(x)$ whose Fourier-Stieltjes transform is $L(u; \sigma_a)$ possesses a continuous derivative of order $m-1$. The proof is based on a quite deep diophantine analysis suggested partially by the work of Ch. Pisot [La répartition modulo un et les nombres algébriques, Ann. Scuola Norm. Super. Pisa (2) 7, 238]. *M. Kac* (Ithaca, N. Y.).

Takahashi, Shin-ichi. Some new properties of Bohr almost periodic Fourier series. Jap. J. Math. 16, 99–133 (1939). [MF 1110]

In the first part the author derives several new theorems concerning derivatives of fractional order for Bohr almost periodic functions. For instance, if $f(x)\sim\sum A_n e^{i\lambda_n x}$, and in addition

$$\int^x f(x)dx=O(|x|^{1-p}), \qquad 0<p\leqq 1,$$

then

$$\sum \operatorname{sgn}\lambda_n\cdot A_n|\lambda_n|^{-q}e^{i\lambda_n x}, \qquad 0<q<p,$$

is again almost periodic. Or, if in addition $f(x)$ satisfies a Lipschitz condition of order p, then

$$\sum \operatorname{sgn}\lambda_n\cdot A_n|\lambda_n|^{q}e^{i\lambda_n x}$$

is again almost periodic. In the second part the author studies the convergence of the "partial sums"

$$S_\sigma(x)=\sum_{|\lambda_n|<\sigma} A_n e^{i\lambda_n x}$$

as σ tends to infinity over a sequence of special values σ_ν, each σ_ν being the center of an interval of length δ_ν which does not contain any exponent λ_n. These investigations were started by the reviewer [Proc. London Math. Soc. 26, 433–452 (1926)] and in some theorems it was assumed that the sequence δ_ν has a positive lower bound. The author introduces weaker assumptions concerning the manner in which δ_ν decreases as σ_ν increases. *S. Bochner.*

Bochner, S. A uniqueness theorem for analytic almost-periodic functions. Duke Math. J. 5, 937–940 (1939). [MF 828]

The following theorem is proved: (i) If each of the functions $f_n(s)=f_n(\sigma+it)$ $(n=1, 2, \cdots)$ is analytic in the strip $\alpha<\sigma<\beta$ and uniformly continuous and almost periodic in the closed strip $\alpha\leqq\sigma\leqq\beta$, (ii) if there exists a constant A such that $M_t|f_n(\sigma+it)|\leqq A$ for $\alpha\leqq\sigma\leqq\beta$ and $n=1, 2, \cdots$ and (iii) if there exists a nonnegative almost periodic function $\varphi(t)$, which does not vanish identically, for which $\lim_{n\to\infty} M_t(|f_n(\alpha+it)|\varphi(t))=0$, then the relation $\lim_{n\to\infty} M_t|f_n(\sigma+it)|=0$ holds for every σ in the open interval $\alpha<\sigma<\beta$. *B. Jessen* (Copenhagen).

Biggeri, Carlos. On abscissas of convergence of the integrals of Laplace and of the Dirichlet series. An. Soc. Ci. Argentina 128, 65–70 (1939). (Spanish) [MF 1081]

The author attempts to prove: "The integrals

$$\int_0^\infty e^{-ts}\varphi(t)dt, \quad \int_0^\infty t^\omega e^{-ts}\varphi(t)dt$$

have always the same abscissa of convergence, ω being an arbitrary complex number." A correlative proposition for Dirichlet series and some corollaries are also given. It may be remarked that the theorem does not hold, for example, for $\omega=-1$. *A. González Domínguez* (Providence, R. I.).

Sewell, W. E. Integral approximation and continuity. Tôhoku Math. J. 46, 75–78 (1939). [MF 1169]

Combining previous results of his own and of Walsh, the author proves: Let C be a rectifiable Jordan curve in the z-plane and $f(z)$ continuous on C. For each n, assume the existence of a polynomial $P_n(z)$ of degree n such that ($k\geqq 0$ integer)

$$\int_C |f(z)-P_n(z)|^p|dz|<n^{-(k+\alpha+1)p}R^{-np},$$

$$0<\alpha\leqq 1;\ p>0;\ R>1.$$

Then $\lim_{n\to\infty} P_n(z)=f(z)$, uniformly on C. Also $f(z)$ is analytic in the open interior of the exterior level-curve C_R and continuous in the closed interior of C_R. Furthermore, $f^{(k)}(z)$ satisfies on C a certain condition of Lipschitz type. *G. Szegö* (Stanford University, Calif.).

Mohan, Brij. Self-reciprocal functions involving Laguerre polynomials. J. Indian Math. Soc. 3, 268–270 (1939). [MF 1024]

The function

$$x^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}L_n^{\frac{1}{2}\nu-\frac{1}{2}}(\tfrac{1}{2}x^2)$$

is shown to be its own J_ν transform: L_n^α is the Laguerre polynomial of order α and degree n. *R. P. Boas, Jr.*

Mohan, B. On self-reciprocal functions. Quart. J. Math., Oxford Ser. 10, 252–260 (1939). [MF 1035]

The function $g(x)=\int_0^\infty P(xy)f(y)dy$ is R_λ (that is, its own J_λ transform) if

$$P(x)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} 2^s G(s)\omega(s)x^{-s}ds,$$

$$f(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{\frac{1}{2}s}G(s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\chi(s)x^{-s}ds,$$

where $G(s)=\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)$; and in the strip S: $a<\sigma<1-a$ $(s=\sigma+it;\ 0<a<\frac{1}{2})$, $\omega(s)=\omega(1-s)$, $\chi(s)=\chi(1-s)$; the lines of integration are in S; and

$$\omega(s)=O\{e^{(\frac{1}{2}\pi-2\alpha+2\eta)|t|}\}, \quad \chi(s)=O\{e^{(\frac{3}{4}\pi-\alpha+\eta)|t|}\},$$

for every positive η and some α in $0<\alpha\leqq\frac{1}{2}\pi$, uniformly in every strip interior to S. Special choices of $f(x)$ and $P(x)$ yield a number of particular R_λ functions and classes of R_λ functions, many of them known. For example,

$$x^{\lambda+2n+\frac{1}{2}}e^{-\frac{1}{2}x^2}{}_1F_1(-n;\ 1+n+\lambda;\ \tfrac{1}{2}x^2), \quad \lambda+2n>-1,$$

is R_λ. *R. P. Boas, Jr.* (Durham, N. C.).

Hartman, Philip. An asymptotic formula for exponential integrals. Amer. J. Math. 62, 115–121 (1940). [MF 962]

The author obtains conditions for the validity of the relation

$$(1) \qquad \int_0^1 f(x)\exp(-sx^\delta)dx\sim\Gamma(1+\delta^{-1})f(0+)s^{-1/\delta},$$

as $|s|\to\infty$. If $f(x)$ is of bounded variation and $\delta>1$, (1) holds uniformly in $|\arg s|\leqq\pi/2$; if $f(x)$ is integrable and $f(0+)$ exists, (1) holds uniformly in any angle $|\arg s|\leqq\pi/2-\epsilon$ $(\epsilon>0)$ if $\delta>1$, and, if $\delta>0$, provided that in addition $|f(x)-f(0+)|\,|\log x|^{1/\delta}\to 0$ as $x\to 0$. If the behavior of $f(x)$ near $x=0$ is suitably restricted, the right side of (1) is the first term of an extended asymptotic relation. For older results, of which these are generalizations in several direc-

tions, and an explanation of their significance, see Wintner, Proc. Nat. Acad. Sci. U. S. A. 20, 57–62 (1934).
R. P. Boas, Jr. (Durham, N. C.).

Moore, Marvin G. On expansions in series of exponential functions. Amer. J. Math. 62, 83–90 (1940). [MF 959]

Expansions of the form

$$f(x)=\sum_{s=1}^{\infty}\sum_{k=1}^{\sigma(s)}P_{ks}(x)\exp(t_{ks}x), \tag{1}$$

associated with an exponential polynomial

$$h(t)=\sum_{m=1}^{N}c_m\exp(a_mt), \qquad N\geqq 2,$$

have been discussed by Carmichael [Trans. Amer. Math. Soc. 35, 1–28 (1933)]; in (1), the numbers t_{ks} are the zeros of $h(t)$, and $P_{ks}(x)$ is a polynomial of degree less than the order of t_{ks}. In Carmichael's work, $f(x)$ was an entire function of exponential type; in the author's, $f(x)$ is analytic in the interior of a convex polygon P' (if P' has interior points), where P' is inside the smallest convex polygon P containing the points a_m, and $f(x)$ is suitably restricted on the boundary of P'. The author's investigation is based on the following representation of the terms in (1):

$$P_{ks}(x)\exp(t_{ks}x)=\frac{1}{2\pi i}\sum_{m=1}^{N}c_m\int_a^{a_m}f(w)dw\int_{C_{ks}}\frac{\exp(a_m+x-w)z}{h(z)}dz, \tag{2}$$

where a is a point of P' and C_{ks} is a small circle about t_{ks}. The representation (2) is a consequence of a biorthogonality relation, namely, that if $f(w)=w^n\exp(t_{jp}w)$, with n less than the order of the zero t_{jp}, then the right side of (2) is equal to 0 or $f(w)$ according as t_{ks} and t_{jp} are different or equal. When $h(t)=e^t-1$, (1) becomes a Fourier series, as was observed by Carmichael; (2) becomes the usual formula for its coefficients; the author's convergence theorem for (1) becomes Jordan's test.
R. P. Boas, Jr. (Durham, N. C.).

Zaanen, A. C. On some orthogonal systems of functions. Compositio Math. 7, 253–282 (1939). [MF 798]

The boundary problem $u''+[\lambda+Q(x)]u=0$,

$$\alpha_{j,1}u(o)+a_{j,2}u'(o)+a_{j,3}u(\pi)+\alpha_{j,4}u'(\pi)=0, \quad j=1,2,$$

with $\pi_{1,2}=\pi_{3,4}$ $(\pi_{i,j}=\alpha_{1i}\alpha_{2j}-\alpha_{1j}\alpha_{2i})$ is discussed. Asymptotic formulas analogous to those of Hobson for the Sturm-Liouville case are obtained, under a variety of assumptions relative to the values of the determinants π_{ij}, $i, j=1, 2, 3, 4$, and with appropriate hypotheses of boundedness of variation or differentiability on $Q(x)$. With the forms obtained the Sturm-Liouville theorems of Haar (equi-convergence), Du Bois-Reymond, and Zygmund (on sets of uniqueness) are extended to the present cases.
R. E. Langer.

Hardy, G. H. Notes on special systems of orthogonal functions (III): A system of orthogonal polynomials. Proc. Cambridge Philos. Soc. 36, 1–8 (1940). [MF 893]

The preliminary notes relate to a function $F(x)$ which is $L^2(0,\infty)$ and which satisfies the functional equation $F(x)=x^{-1}F(x^{-1})$. The Mellin transform of $F(x)$ is $f(s)=\pm k(s)f(1-s)$, where $|k(\frac{1}{2}+it)|$ is unity. If $f(s)=l(s)g(s)$, where $l(s)=k(s)l(1-s)$, then $g(s)$ is an even or odd function of $s-\frac{1}{2}$. The case when $g(s)=(s-\frac{1}{2})^n$ is used to construct an eigenfunction with eigenvalue $(-)^n$ for a singular kernel $K(x)$ over the range $(0,\infty)$. The relation

$$2\pi\int F(x)G(x)dx=\int f(\tfrac{1}{2}+it)g(\tfrac{1}{2}+it)dt$$

between two real functions $F(x)$, $G(x)$ and their Mellin transforms $f(s)$, $g(s)$ is then used to construct corresponding systems of orthogonal functions both of which are complete when $l(\frac{1}{2}+it)=O[\exp(-\delta|t|)]$. The completeness of the set of polynomials of Laguerre is proved in this way and a brief discussion is given of a related set of polynomials $p_n(s-\frac{1}{2})$ with the generating function $(1-w)^{s-1}\cdot(1+w)^{-s}$. The reviewer notes that in the definite integral (5.2) for $p_n(s-\frac{1}{2})$, the index of $(\tan\frac{1}{2}\theta)^{s-\frac{1}{2}}$ should not be $s-\frac{1}{2}$ but $s-1$ for $p_n(s-\frac{1}{2})=\frac{1}{2}g_n(-s)+\frac{1}{2}g_n(1-s)$, where $g_n(z)$ is the polynomial of Mittag-Leffler, and, if $|z|<1$, this polynomial has two representations

$$\pi g_n(z)=2\int_0^{\pi}(\cot\tfrac{1}{2}u)^zT(\tfrac{1}{2}\pi z)T(nu)du,$$

where the function $T(x)$ is either a sine or a cosine. A set of functions $q_n(x)$ related to the set $p_n(s-\frac{1}{2})$ is orthogonal over the range $(-\infty,\infty)$ with weight factor sech (πx). The paper ends with a determination of a similar set of functions with weight factor sech^2 (πx).
H. Bateman.

Hille, Einar. Sur les séries associées à une série d'Hermite. C. R. Acad. Sci. Paris 209, 714–716 (1939). [MF 802]

Hille, Einar. Contributions to the theory of Hermitian series. Duke Math. J. 5, 875–936 (1939). [MF 827]

Let

$$f(z)=\sum_0^{\infty}f_nh_n(z), \quad h_n(z)=\exp(-\tfrac{1}{2}z^2),$$

$$H_n(z)=(-1)^n\exp(\tfrac{1}{2}z^2)\frac{d^n}{dz^n}(e^{-z^2}).$$

Let $A_{2m}=|H_{2m}(0)|$, $A_{2m+1}=|H'_{2m+1}(0)|(4m+3)^{-\frac{1}{2}}$, $c_n(z)=A_n\cos(\nu_nz-\frac{1}{2}n\pi)$, $s_n(z)=A_n\sin(\nu_nz-\frac{1}{2}n\pi)$, $\nu_n=(2n+1)^{\frac{1}{2}}$. Observing that $h_n(z)$ is the solution of the integral equation

$$w(z)=c_n(z)+\nu_n^{-1}\int_0^z t^2\sin\nu_n(z-t)w(t)dt,$$

the author develops in chapter 1 various asymptotic expressions for $h_n(z)$. He points out that analogues of some of these expressions have been obtained by Szegö in his recent book on orthogonal polynomials [Amer. Math. Soc. Colloquium Publ. 23, 1939]. In chapter 2 it is proved that the domain of convergence of the series $\sum f_nh_n(z)$ is the horizontal strip $|y|<\tau$, with possible addition of points on the lines $y=\pm\tau$, where $\tau=-\limsup\nu_n^{-1}\log(A_n|f_n|)$. It is also proved that the series $\sum f_nh_n(z)$ converges for a nonreal z if and only if the associated series $C(z)=\sum f_nc_n(z)$ converges for this value. In chapter 3 it is observed that $\delta_zh_n(z)=(2n+1)h_n(z)$, where $\delta_z=z^2-d^2/dz^2$, and differential operators of the form $G(\delta_z)$ are discussed. Among the various results obtained here we mention only the following two. Let $E(\rho,T)$ be the class of entire functions of order ρ and type T. Let σ be defined from $1/\rho+1/2\sigma=1$ for $\rho>2$; $\sigma=\frac{1}{2}$ if $\rho=\infty$; $\sigma=1$ if $0\leqq\rho\leqq 2$. Then, if $f\varepsilon E(\rho,T)$, $\rho>2$, we have

$$\limsup_{k\to\infty}k^{-1/\sigma}|\delta_z^kf(z)|^{1/k}\leqq(2/e)^{1/\sigma}(\rho T)^{2/\rho};$$

if $f\varepsilon E(2,T)$ the right-hand member of the previous in-

equality should be replaced by $A+BT$, where A and B are suitable constants, and if $f\epsilon E(\rho, T)$, $0\leqq\rho<2$, it should be replaced by C, where C is an absolute constant such that $1/e\leqq C\leqq 2^{3/2}$. We say that the differential operator $G(\delta_z)=\sum_0^\infty g_k\delta_z^k$ is applicable to the functions $f(z)$ if the series $G(\delta_z)f(z)=\sum_0^\infty g_k\delta_z^k f(z)$ converges at every point where $f(z)$ is holomorphic. Then a necessary and sufficient condition that $G(\delta_z)$ shall be applicable to every $f\epsilon E(\rho\leqq\rho_1, T<\infty)$ is that $G(w)\epsilon E(\rho<\sigma_1)+E(\sigma_1, 0)$ provided $\rho_1\geqq 2$. If $\rho_1<2$ the condition is merely sufficient, but it is necessary that $G(w)\epsilon E(\rho\leqq 1, T<1)$.

In chapter 4 the author discusses various conditions on the sequence of factors $\{a_n\}$ under which the function $\sum a_n f_n h_n(z)$ can be analytically continued along any path C from $z=0$, along which $f(z)$ can be so continued. In chapter 5, being guided by the close analogy between the Hermitian series on one hand, and Dirichlet and power series on the other, the author discusses the singularities of the series $\sum f_n h_n(z)$ on the boundary of the strip of convergence. Among various other results it is proved that this series has a singular point at $\pm\tau i$ according as $(\pm i)^n f_n\geqq 0$ for all large values of n, provided $\tau>0$. Among various gap-theorems it is proved in this chapter that the series $\sum_{k=1}^\infty f_{n_k}h_{n_k}(z)$ has its strip of convergence as natural domain of existence if $n_k k^{-2}\rightarrow\infty$ and $\liminf[(n_{k+1})^{\frac12}-(n_k)^{\frac12}]>0$. In chapter 6 the author studies the relationships between the Hermitian series $\sum f_n h_n(z)$ and its associated Fourier series $C(z)=\sum f_n c_n(z)$, $S(z)=\sum f_n s_n(z)$, and associated Dirichlet series $E^+(z)=\sum f_n e_n^+(z)$, $E^-(z)=\sum f_n e_n^-(z)$, where $e_n^\pm(z)=A_n(\mp i)^n\exp(\pm i\nu_n z)$. He derives an expansion of $f(z)$ in terms of $E^\pm(z)$ and shows that $f(z)$ is holomorphic and is represented by this expansion in the cross section of the Mittag-Leffler principal star of $C(z)$ and of its reflection in the origin. It is also shown that there exist Hermitian series (in fact even of entire functions) which have no singular points on the boundary of their strip of convergence. Finally, it is proved that the four associated functions are holomorphic in every circle $|z|<R$, where $f(z)$ is holomorphic, and, if z_0 is a singular point of $C(z)$ which is nearest to the origin, then z_0 or $-z_0$ is a singular point of $f(z)$, while every singular point of $f(z)$ is a singular point of either $C(z)$ or $C(-z)$. *J. D. Tamarkin.*

Hille, Einar. Contributions to the theory of Hermitian series. II. The representation problem. Trans. Amer. Math. Soc. 47, 80–94 (1940). [MF 1071]

Let

$$H_n(z)=[\pi^{\frac12}2^n n!]^{-\frac12}(-1)^n e^{z^2/2}\frac{d^n}{dz^n}(e^{-z^2})$$

be the nth normalized orthogonal function of Hermite. If $f(x)$ is a measurable function such that

$$x^n\exp(-x^2/2)f(x)\epsilon L(-\infty,\infty)$$

for all $n=0, 1, 2, \cdots$, let

$$\sum_0^\infty f_n H_n(x),\quad f_n=\int_{-\infty}^\infty f(t)H_n(t)dt$$

be the associated Fourier-Hermite series. The author establishes the following results:

1. Let $f(z)$ be analytic. A necessary and sufficient condition that the Fourier-Hermite series shall converge to $f(z)$ in the strip S_τ: $-\tau<y<\tau$ is that $f(z)$ is holomorphic in S_τ and that to every β, $0\leqq\beta<\tau$, there exists a finite positive $B=B(\beta)$ such that

$$|f(x+iy)|\leqq B\exp[-|x|(\beta^2-y^2)^{\frac12}],\quad -\infty<x<\infty;\ -\beta\leqq y\leqq\beta.$$

2. Let $f(z)$ be holomorphic in the half-plane $y>-\alpha$, $\alpha>0$, and let

$$\limsup_{r\to\infty}(1/r)\log|f(re^{i\theta})|\leqq M,\qquad 0\leqq\theta\leqq\pi.$$

The Fourier-Hermite expansion of $f(z)$ can never converge outside of the real axis.

3. Let $f(z)$ be meromorphic in $y>-\alpha$. Let its zeros and poles in this half-plane be $a_1, a_2, \cdots; b_1, b_2, \cdots$, respectively, where, in addition, $\Im(b_n)\geqq\beta>0$. Put $\arg a_n=\alpha_n$, $\arg b_n=\beta_n$, $a(r)=\sum\sin\alpha_n$, $b(r)=\sum\sin\beta_n$, where the summations extend over all zeros and poles, respectively, of absolute value between β and r. Let

$$d=\limsup_{r\to\infty}(1/r)[b(r)-a(r)]$$

and

$$q=\limsup_{r\to\infty}(r\log r)^{-1}\int_0^\pi\log|f(re^{i\theta})|\sin\theta d\theta.$$

If $f(z)$ has a Fourier-Hermite expansion whose ordinate of convergence equals τ, then $\tau\leqq\min(\alpha,\beta,q+\pi d)$. It is shown by examples that this result is the best possible.

J. D. Tamarkin (Providence, R. I.).

Madhava Rao, B. S. Quantum-mechanical interpretation of a result concerning Hermite polynomials. Proc. Indian Acad. Sci., Sect. A. 10, 217–219 (1939). [MF 693]

Quantentheoretischer Interpretation der Formel

$$\lim_{n\to\infty}\frac{1}{2^n n!}\int_\alpha^\beta\{H_n(x)\}^2dx=0,$$

wo

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n(e^{-x^2})}{dx^n}.$$

C. S. Meijer (Groningen).

Thiruvenkata Char, V. R. Note on some formulae involving the Laguerre and Hermitian polynomials and Bessel functions. Proc. Indian Acad. Sci., Sect. A. 10, 229–234 (1939). [MF 687]

Berechnung verschiedener bekannter Integrale, zum Beispiel,

$$\int_{-\infty}^\infty e^{-sx^2}H_{2n}(x)dx=\pi^{\frac12}\frac{(2n)!}{n!}\frac{(1-s)^n}{s^{n+\frac12}},$$

$$\int_0^\infty e^{-t}t^{(n+\alpha)/2}L_n^{(\alpha)}(t)J_{\alpha-n}(2(xt)^{\frac12})dt=e^{-x}x^{(n+\alpha)/2}L_n^{(\alpha)}(x).$$

Ableitung der bekannten erzeugenden Funktionen von $L_n^{(\alpha)}(x)$; Entwicklungen, zum Beispiel,

$$e^{x^2}\cos 2xt=\sum_{n=0}^\infty(-1)^n\frac{H_{2n}(t)}{(2n)!}x^{2n}.$$

C. S. Meijer (Groningen).

Iyengar, K. S. K. A new proof of Mehler's formula and other theorems on Hermitian polynomials. Proc. Indian Acad. Sci., Sect. A. 10, 211–216 (1939). [MF 692]

Elementarer Beweis der Mehlerschen Formel

$$\sum_{n=0}^\infty t^n\psi_n(x)\psi_n(y)=\frac{1}{(\pi(1-t^2))^{\frac12}}\exp\left\{\frac{2xyt}{1-t^2}-\left(\frac{x^2+y^2}{2}\right)\left(\frac{1+t^2}{1-t^2}\right)\right\},$$

wo

$$\psi_n(x)=\frac{(-1)^n e^{\frac{1}{2}x^2}\dfrac{d^n}{dx^n}(e^{-x^2})}{(2^n n!\pi^{\frac{1}{2}})^{\frac{1}{2}}}.$$

Abschätzungen für $\{\psi_n(x)\}^2$ und $\sum_{r=1}^n\{\psi_r(x)\}^2$.
C. S. Meijer (Groningen).

Iyengar, K. S. K. A new proof of the formula for the generating function of Laguerre polynomials and other related formulae. Proc. Indian Acad. Sci., Sect. A. **10**, 181–185 (1939). [MF 691]

Elementarer Beweis der Beziehung

$$\sum_{n=0}^{\infty} t^n L_n(x)L_n(y)=(1-t)^{-1}\exp\left(\frac{(x+y)t}{t-1}\right)I_0\left(\frac{2(xyt)^{\frac{1}{2}}}{1-t}\right).$$

Ferner wird gezeigt

$$\lim_{n\to\infty}\left[\sum_{r=1}^{n}\{L_r(x)\}^2/n^{\frac{1}{2}}\right]=e^x/(\pi x^{\frac{1}{2}}).$$

C. S. Meijer (Groningen).

Sastry, B. S. Note on a type of generalised Laguerre polynomial. Proc. Indian Acad. Sci., Sect. A. **10**, 176–180 (1939). [MF 690]

Verfasser betrachtet das Polynom

$$\pi_n(x)=e^x\frac{d^n}{dx^n}\{e^{-x}A_n(x)\};$$

hierin ist $A_n(x)$ ein Polynom n-ten Grades in x, das gewissen Voraussetzungen genügt. Er leitet Entwicklungen, rekurrente Beziehungen und erzeugende Funktionen für $\pi_n(x)ab$. *C. S. Meijer* (Groningen).

Dickinson, D. R. On Tchebycheff polynomials. Quart. J. Math., Oxford Ser. **10**, 277–282 (1939). [MF 1038]

Let $\phi_0(x), \cdots, \phi_n(x)$ be continuous in $[a, b]$. An expression $\sum_0^n A_k\phi_k(x)$ is called a T-polynomial if it can have no more than n distinct roots in $[a, b]$ if not all real coefficients A_k are zeros. The functions $\{\phi_k\}$ are then said to constitute a T-system of order n. It is well known that a necessary and sufficient condition for this is that the determinant $|\phi_i(x_j)|_{i,j=0}^n\neq 0$. A point $\bar{x}$ in $[a, b]$ is called a simple or a double root of a T-polynomial according as the polynomial changes, or does not change, the sign when x passes through $\bar{x}$. The end-points a, b, in case they are roots, are considered as simple roots. If s is the number of simple roots and d the number of double roots, it is well known that $s+2d\leqq n$. The author proves that, given $s+d$ distinct points $x_1, \cdots, x_s$ and $y_1, \cdots, y_d$, such that $a\leqq x_i\leqq b$, $a<y_k<b$, it is always possible to construct a T-polynomial having simple roots at x_i and double roots at y_k. This theorem was also given by S. Bernstein [Extremal Properties of Polynomials (in Russian), 1937, p. 9]. The author states, however, that Bernstein's proof does not seem to be complete, and gives a proof on different lines.
J. D. Tamarkin (Providence, R. I.).

Shohat, J. Application of orthogonal Tchebycheff polynomials to Lagrangean interpolation and to the general theory of polynomials. Ann. Mat. Pura Appl. **18**, 201–238 (1939). [MF 920]

The author derives several theorems on the distribution of the roots of orthogonal polynomials and on convergence and mean convergence of the Lagrange interpolation polynomials taken at the roots of the orthogonal polynomials. An important tool which he uses is the so-called "second orthogonality property of orthogonal polynomials" discovered by Tchebycheff. He also obtains an estimate of an arbitrarily given linear combination of the coefficients of any polynomial $f_n(x)$ of degree n if we know its upper bound at $n+1$ properly specified points. *P. Erdös.*

Dieulefait, C. On the Poisson-Charlier series. An. Soc. Ci. Argentina **128**, 10–24 (1939). (Spanish) [MF 1079]

Let $f(v)\geqq 0$ be defined for all integral values of $v\geqq 0$, and consider its Charlier series

$$(1)\qquad \psi_0(k, v)\sum_{n=0}^{\infty} a_n p_n(k, v),$$

where $\psi_0(k, v)=(e^{-k}k^v)/v!$, $p_n(k, v)$ is the nth Charlier polynomial and

$$a_n=\frac{k^n}{n!}\sum_{t=0}^{\infty} f(t)p_n(k, t).$$

The present paper contains: (1) [paragraphs 1–13] a presentation of known results about Charlier polynomials (orthogonality, generating function, etc.); (2) [paragraphs 13–14] the two following theorems. I. The series (1) converges to $f(v)$ if, for a sufficiently small ϵ,

$$\sum_{t=1}^{\infty}\frac{f(t)\epsilon^t}{t\psi_0(k, t)}<\infty.$$

II. The series (1) converges to $f(v)$ if, μ, η, α being positive numbers, $f(t)<\mu\eta^t t\psi_0^{\alpha+1}(k, t)$ for t sufficiently large. It may be remarked that these results are rather particular cases of already known simpler theorems. I follows from Jacob's theorem [Giorn. Ist. Ital. Attuari **4**, 230 (1933)]; and II is contained in a theorem of Obrechkoff [Actual. Sci. Ind. 740, 44, Theorem II].
A. González Domínguez (Providence, R. I.).

Cassina, Ugo. Formole sommatorie e di quadratura ad ordinate estreme. Ist. Lombardo, Rend. **72**, 225–274 (1939). [MF 925]

Cassina, Ugo. Estensione del teorema di Rolle al calcolo delle differenze ed applicationi. Ist. Lombardo, Rend. **72**, 323–332 (1939). [MF 929]

1. Between the terms of a sequence of real numbers $f_0, f_1, \cdots, f_n$ and their differences of various orders there are many particular linear relations with constant coefficients which are of interest. The author derives such relations, or summation formulae, which are useful in computing approximately the sum $S=f_0+f_1+\cdots+f_n$ in terms of a few only of the f_r. A typical result is the relation

$$(1)\qquad \sum_{r=0}^{n} f_r=\sum_{s=0}^{m-1} A_s(\Delta^s f_{n-s}+(-1)^s\Delta^s f_0),\qquad 2m-2<n,$$

where

$$A_s=(-1)^s\binom{2m-s-1}{m}\binom{n+1}{s+1}(m!)^2/(2m)!,$$

which is exact provided $f_r=P(r)$ $(0\leqq r\leqq n)$, $P(x)$ being a polynomial of degree less than $2m$. This suggests that (1) will become an identity provided a certain linear combination of the $\Delta^{2m}f_s$ $(s=0, 1, \cdots, n-2m)$ is added to the right hand side of (1). The author determines this "remainder" in this and similar situations by means of the following elementary theorem: Let $Rf=c_0f_0+\cdots+c_nf_n$ be a linear

form in the f_r, considered as a linear operator operating on the sequence f_r. If $Rf=0$ for any sequence of the form $f_r=P(r)$ $(r=0, \cdots, n)$, where $P(x)$ is a polynomial of degree less than k $(\leqq n)$, then $Rf=\sum_{s=0}^{n-k}\gamma_s\Delta^k f_s$, the coefficient γ_s being the value of R operating on the sequence

$$\binom{r-s-1}{k-1}, \quad r=0, \cdots, n; \quad \binom{t}{k-1}=0 \quad \text{if } t<k-1.$$

Now the difference between the two sides of (1) is such an operator enjoying the required property for $k=2m$. It is thus found that (1) becomes an identity in the f_s provided the remainder term

$$R=(-1)^m\frac{(m!)^2}{(2m)!}\sum_{s=0}^{n-2m}\binom{n-s-m}{m}\binom{m+s}{m}\Delta^{2m}f_s$$

is added to its right hand side. Moreover, since the coefficients of $\Delta^{2m}f_s$ are of the same sign, we may also write

$$R=(-1)^m\frac{(m!)^2}{(2m)!}\binom{n+1}{2m+1}\Delta_{2m}, \tag{3}$$

where Δ_{2m} is between the least and largest term of the sequence $\Delta^{2m}f_s$ $(0\leqq s\leqq n-2m)$. This result is an improvement on Lubbock's formula [Steffenson, Interpolation, Baltimore, §15] which is exact only for sequences arising from polynomials of degree not greater than m. For the many other results of similar type we refer to the paper. Finally, from each summation formula a corresponding quadrature formula is derived by direct passage to the limit.

2. The real sequence $f_0, f_1, \cdots, f_m$ (considered as an arithmetical function defined over the range $0 \cdots m$) is said to have a zero at a if $f_a=0$, or a variation at a provided $f_af_{a+1}<0$. By a natural adaptation of the arguments leading to Rolle's theorem of the calculus, the author establishes its following analogue in the calculus of finite differences: If the sequence f_r has zeros or variations (in any of the four possible combinations) at $r=a$ and $r=b$ $(a<b)$, then the sequence Δf_r has a zero or a variation in the range $a \cdots (b-1)$. This theorem leads readily to the following analogue of the formula of polynomial interpolation: If $0\leqq x_1<x_2<\cdots<x_n<m$ is a sequence of integers and x another integer not less than 0, less than m, and if $F(x)$ is the polynomial of degree not greater than $n-1$ satisfying $F(x_\nu)=f_{x_\nu}$ $(\nu=1, \cdots, n)$, then $f_x=F(x)+(x-x_1)(x-x_2)\cdots(x-x_n)\Delta_n/n!$, where Δ_n is a number between the least and largest among $\Delta^n f_s$ $(s=a, a+1, \cdots, b-n+1)$, while $a=\min(x, x_\nu)$, $b=\max(x, x_\nu)$. An application of this result leads elegantly to the differential expression (3) (see above) for the remainder of some of the summation formulae established in the first paper. *I. J. Schoenberg.*

de Losada y Puga, Cristóbal. The interpolation formula of Stirling deduced from the Taylor's series. Revista Univ. Católica Peru **7**, 177–186 (1939). (Spanish) [MF 980]

Differential and Integral Equations, Functional Analysis, Ergodic Theory

Rainville, E. D. Adjoints of linear differential operators. Amer. Math. Monthly **46**, 623–627 (1939). [MF 975]

The author defines an operator α such that for any linear differential operator A, αA is the adjoint of A. He proves that $\alpha^2A=A$ and that $(\alpha A_1)(\alpha A_2)=\alpha(A_1A_2)$. In case that A is of second order, the author finally considers a function of the coefficients of A which remains invariant when A is replaced by the adjoint αA. *E. Rothe.*

Mambriani, Antonio. Sugli algoritmi integro-differenziali lineari. Ist. Lombardo, Rend. **72**, 147–159 (1939). [MF 947]

For an ordinary linear differential equation of the second order the author gives a formal process of solution which in a particular instance reduces to the usual method of successive approximations. In only this latter case does he consider the convergence of the process to a solution of the given equation. The algorithm presented is an immediate consequence of elementary properties of differential equations and the well-known factorization of a linear differential operator in terms of an associated fundamental set of solutions [see, for example, Ince, Ordinary Differential Equations, pp. 119–120]. *W. T. Reid* (Chicago, Ill.).

Whyburn, W. M. Over and under functions as related to differential equations. Amer. Math. Monthly **47**, 1–10 (1940). [MF 1150]

This paper gives a brief exposition of some facts about over and under functions with special emphasis on their use in the definition of the Perron integral, in the solution of differential equations of type $y'=f(x, y)$ (including existence of minimal and maximal solutions) and in the treatment of the Dirichlet problem (connection with super- and sub-harmonic functions). It is indicated that the use of over and under functions constitutes an extension of the idea of the Dedekind-cut. *W. J. Trjitzinsky.*

Palama, Giuseppe. Sulle equazioni differenziali lineari soddisfatte dal prodotto di integrali particolari di due equazioni differenziali lineari omogenee assegnate e su alcune formule integrali dei polinomi di Laguerre e di Hermite. Ann. Mat. Pura Appl. **18**, 309–325 (1939). [MF 924]

Dette (u_1, u_2) e (v_1, v_2) due coppie di integrali indipendenti rispettivamente delle equazioni

$$u''=f_{01}u'+f_{02}u, \quad v''=\varphi_{01}v'+\varphi_{02}v,$$

l'autore determina l'equazione differenziale che ha per integrale generale

$$c_1u_1v_1+c_2u_1v_2+c_3u_2v_1+c_4u_2v_2$$

e ciò valendosi di un metodo che è applicabile anche per risolvere il problema analogo relativo ad una coppia di equazioni di ordine superiore al secondo. Come applicazione l'autore scrive le equazioni differenziali soddisfatte dal prodotto di due polinomi di Hermite o di Laguerre e se ne serve per dimostrare alcune identità integrali relative a tali funzioni. *C. Miranda* (Torino).

Rádl, Franz. Über die Teilbarkeit des gewöhnlichen Differentialpolynoms dritter Ordnung durch ein ähnliches Polynom zweiter Ordnung. Math. Z. **45**, 719–734 (1939). [MF 1409]

Illustrating his theory of generalized division for linear differential polynomials [these Rev. **1**, 15 (1940)], the author derives conditions for the divisibility of a polynomial of the third order by one of the second. *J. F. Ritt* (New York, N. Y.).

Schouten, J. A. und van der Kulk, W. Beiträge zur Theorie der Systeme Pfaffscher Gleichungen. I. Ein Theorem über die Klassen der Faktoren eines Systems. Nederl. Akad. Wetensch., Proc. 43, 18–31 (1940). [MF 1235]

Proof for the cases $q=n-2$, $n-3$, $n-4$ of the theorem: Every system of q Pfaff equations in n variables is equivalent to a system of q equations of class not greater than p for even p and of class not greater than $p+1$ for odd p ($p=n-q$). The proof for $q=n-5$ and $q=n-6$ is promised for the second communication. For the other cases the theorem remains a hypothesis. *D. J. Struik.*

Chern, Shing-Shen. Sur la géométrie d'un système d'équations différentielles du second ordre. Bull. Sci. Math. 63, 206–212 (1939). [MF 365]

Following up a remark of E. Cartan concerning a paper of Kosambi [Math. Z. 37, 603 (1933)], this paper makes a geometrical study of the system of differential equations

$$(1)\qquad \frac{d^2x^i}{dt^2}+F^i\left(\frac{dx^i}{dt};x^i;t\right)=0,\qquad i=1,\cdots,n.$$

We seek the geometrical properties associated with the system (1) and which are characterized intrinsically by either of the following groups of transformations

$$(A)\quad \bar{x}^i=f^i(x^i),\ \bar{t}=t;\qquad\qquad (B)\quad \bar{x}^i=f^i(x^i,t),\ \bar{t}=t.$$

The problem is immediately reduced to the equivalence of systems of pfaffians. One finds that in the case of either group of transformations it is possible to define an affine connection in the space of variables dx^i/dt; x^i, t and the tensors of curvature and torsion are computed. In problem (A) the case where F^i is homogeneous of the second degree in the dx^i/dt is of especial interest. *E. W. Titt.*

Germay, R. H. J. Extension, aux systèmes complètement intégrables d'équations aux différentielles totales, d'un théorème de M. É. Picard relatif aux systèmes normaux d'équations différentielles. Mathesis 53, 216–222 (1939). [MF 852]

Consider the system of total differential equations

$$(1)\qquad \frac{\partial z_i}{\partial x^k}=a_{ik}(x^1,\cdots,x^n;z^1,\cdots,z^m).$$

This paper is concerned with the following property: In the neighborhood of a point x_0^i, z_k^0 at which the a_{ik} are analytic the only solution of the completely integrable system (1) which takes the values z_k^0 at x_0^i is the analytic one. The method is an extension of that used by Picard [Traité d'analyse, 1893, tome 2, 316] to prove the corresponding result for ordinary equations. *E. W. Titt.*

Giraud, Georges. Sur une classe d'équations linéaires ou figurent des valeurs principales d'intégrales simples. Ann. École Norm. 56, 119–172 (1939). [MF 384]

A function $f(x)$ is said to be of class $C_h^{(n)}$ ($n\geqq 0$, $h>0$) if it is continuous, together with its nth partial derivative, which latter satisfies a uniform Hölder condition with exponent h. We consider an abstract, compact, one dimensional, Riemannian manifold V of class $C_h^{(1)}$, that is, one (1) which consists of a finite number of simple closed curves each of which possesses representations in preferred coordinate systems x by periodic transformations $X=X(x)$, two such systems x and y being connected by a transformation $y=y(x)$ in which $y(x)$ is of class $C_h^{(1)}$ with $y'(x)>0$ and periodic, and (2) on which is defined a covariant length vector $\omega(x)$ of class $C_h^{(0)}$ in any preferred coordinate system. Points of V are denoted by capitals and their coordinates by small letters. We define $ds_X=\omega(x)dx$ and $|X-\Xi|$ as the smallest distance from X to Ξ along V, provided X and Ξ are on the same closed curve.

The author studies integral equations of the form

$$(1)\qquad g(X)u(X)-\lambda\textstyle\int_V G(X,A)u(A)ds_A=f(X),$$

in which $g(X)$ and $f(X)$ are of class $C_\alpha^{(0)}$ for some α and a solution $u(X)$ of the same type is sought. The function $G(X,\Xi)$ is assumed to be of class $C_h^{(0)}$ in both X and Ξ if X and Ξ are on different curves; when X and Ξ are on the same curve, $G(X,\Xi)$ is assumed to be of the form

$$G(X,\Xi)=\frac{\pi}{\omega}c(x)\cot\frac{\pi(x-\xi)}{\omega}+G_2(X,\Xi),$$

where ω is the period of the coordinate system, $c(x)$ is a contravariant tensor of class $C_h^{(0)}$, and $G_2(X,\Xi)$ satisfies the conditions

$$|G_2(X,\Xi)|\leqq K_1|X-\Xi|^{h-1},$$
$$|G_2(X,\Xi)-G_2(Y,\Xi)|\leqq K_2|X-Y|^h|A-\Xi|^{-1},$$
$$|G_2(X,\Xi)-G_2(X,H)|\leqq K_3|\Xi-H|^h|B-X|^{-1},$$

where A is the nearer one of X and Y to Ξ and B is the nearer of Ξ and H to X. The integral in (1) is assumed to be the principal value as defined by

$$\int_V G(X,A)u(A)ds_A=\lim_{\substack{\sigma\to 0\\ \tau\to 0}}\int_{V-\delta_{X,\sigma,\tau}}G(X,A)u(A)ds_A,\qquad \lim_{\substack{\sigma\to 0\\ \tau\to 0}}(\tau/\sigma)=1,$$

where $\delta_{X,\sigma,\tau}$ denotes an interval $x-\sigma<a<x+\tau$. It is also assumed that the scalar functions $g(X)$ and $c(x)\omega(x)$ do not vanish simultaneously.

Let C denote the set of all (complex) numbers λ of the form $\lambda=\pm(ig(X))/[\pi c(x)\omega(x)]$, it being understood that ∞ is in C if and only if $c(x)\omega(x)=0$ for some x. The author demonstrates the following results: Let D be any domain which contains no points of C. Then there exist two non-negative integers r and s and an isolated set E in D, all depending only on D, such that if λ is in $D-E$, the homogeneous form of (1) has exactly r linearly independent solutions and that of the associated equation

$$(2)\qquad g(X)u(X)-\lambda\textstyle\int_V G(A,X)u(A)ds_A=f(X)$$

has exactly s linearly independent solutions. If P is a point of E, there exists an integer $t_p>0$ such that the homogeneous forms of (1) and (2) have exactly $r+t_p$ and $s+t_p$ solutions, respectively. In order that the nonhomogeneous equation (1) have a solution, it is necessary and sufficient that $f(X)$ be orthogonal to all the solutions of the homogeneous form of (2). If D contains the point $\lambda=0$, then $r=s=0$ and if D contains $\lambda=\infty$, then $r=s$, but neither is necessarily zero. For $\lambda=\infty$, we understand that (1) has the form

$$\textstyle\int_V G(X,A)u(A)ds_A=f(X).$$

Examples are given to show that, for the general D, it is not necessarily true that $r=s$ or that either is zero. Applications to boundary value problems for elliptic partial differential equations are suggested. *C. B. Morrey, Jr.*

Goldfain, I. Sur un cas particulier d'une équation intégrale linéaire de Fredholm à noyau non symétrique. Rec. Math. (Moscou) 6 (48), 149–159 (1939). (Russian. French summary) [MF 1439]

The author demonstrates that the essential properties of "regular" symmetric kernels will hold for kernels $K_n(x, y) = \sum a_{ij}\alpha_i(x)\alpha_j(x)$ $(i, j=1, \cdots, n)$, where the constants a_{ij} are real or complex and the $\alpha_i(x)$ form a system of functions orthonormal on an interval (a, b). It is supposed that the determinant $|a_{ij}| \neq 0$. The characteristic values of K_n satisfy an equation $D(\lambda)=0$, where $D(\lambda)$ is a certain determinant. The author assumes that, when some of the roots of $D(\lambda)=0$ are coincident, all the elementary divisors are linear. The significance of this paper is increased by the author's remark to the effect that if $K(x, y)$ is measurable, nonsymmetric and such that $\int_a^b\int_a^b K(x, y)^2 dxdy$ is finite, then $K(x, y)$ can be approximated in the mean square by kernels of the form $K_n(x, y)$. *W. J. Trjitzinsky* (Urbana, Ill.).

Goldstine, Herman H. Minimum problems in the functional calculus. Bull. Amer. Math. Soc. 46, 142–149 (1940). [MF 1262]

In this paper the author first gives the obvious necessary conditions and sufficient conditions, in terms of the first and second differentials, that a real-valued function defined on an open set in a normed linear space should have a minimum. Most of the paper is taken up with the more difficult problem of determining such conditions when the class of admissible points is required to satisfy an equation of an abstract functional character. Various hypotheses are described under which a multiplier rule can be obtained. Sufficient conditions for a minimum are also given.
L. M. Graves (Chicago, Ill.).

Godefroy, Marcel. Sur l'extension des systèmes différentiels aux espaces métriques. C. R. Acad. Sci. Paris 209, 593–595 (1939). [MF 520]

Associated with a differential equation $dx/dt=f(x, t)$ in real variables is a one-parameter family of transformations $P'(x', t')=\tau_h P(x, t)$ in an auxiliary real parameter h: $t'=t+h$, $x'=x+hf(x, t)$. For $P(x, t)$ fixed, the point $P'=\tau_h P$ describes a half-line $\Delta(P)$ as $h>0$ varies. A right integral of τ_h on the interval (t_0, t_1) is a continuous curve $x(t)$ admitting $\Delta(P)$ as a right half-tangent for $t_0 \leqq t \leqq t_1$. The author generalizes these notions by replacing the straight lines $t=$const. by point sets E_t so metrized as to form a Fréchet metric space. Sufficient conditions for the existence and unicity of generalized right integrals are announced without proofs. The investigations are related to some earlier studies of Montel [Bull. Sci. Math. (2) 50, 205–217 (1926)] and Zaremba [Bull. Sci. Math. (2) 60 (1936)].
A. D. Michal (Pasadena, Calif.).

Kakutani, S. Some characterizations of Euclidean space. Jap. J. Math. 16, 93–97 (1939). [MF 534]

The central results of this paper are: (1) If E is a Banach space such that there exists a projection with norm one on every closed linear subspace G of E, then E is unitary, that is, the norm satisfies the equation $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$. (2) If, for every G, we can find a method of extending the linear functionals defined on G to the full space so that the method is both linear and norm preserving for the functionals, then E is unitary. The assumption that the dimensionality of E is at least three is made. The problem of extending a linear transformation defined on G to all E is considered and it is also proved that the quasi-convex spaces of Kantorovitch [C. R. Acad. Sci. U.R.S.S. 1, 207–210 (1935)] are unitary. *F. J. Murray.*

Dirac, P. A. M. A new notation for quantum mechanics. Proc. Cambridge Philos. Soc. 35, 416–418 (1939). [MF 546]

The author proposes using the symbol $\rangle$ instead of the letter ψ to represent a Hilbert space vector, and $\langle$ to stand for its conjugate $\bar\psi$. Vectors with subscripts, such as ψ_a and $\bar\psi_a$, are to be denoted by $|a\rangle$ and $\langle a|$. The scalar products $\bar\psi\psi$, $\bar\psi\psi_a$, $\bar\psi_a\psi$ and $\bar\psi_a\psi_b$ are replaced by $\langle\,\rangle$, $\langle|a\rangle$, $\langle a|\rangle$ and $\langle a|b\rangle$. The eigenvectors $\psi(q')$ corresponding to the eigenvalues q' of an observable q are denoted by $|q'\rangle$, treating the eigenvalue q' as a label. For a basic set of eigenvectors, the coordinate or representative of an arbitrary vector $|a\rangle$ corresponding to q' becomes in this notation $\langle q'|a\rangle$. Likewise, if α is a linear operator, its matrix representative in terms of the basic vectors is denoted by $\langle q'|\alpha|q''\rangle$. The notation has the advantage that the symbol for a vector or operator is similar to the symbol for the corresponding representative. The author suggests "bra" and "ket" as names for $\langle$ and $\rangle$. *O. Frink* (State College, Pa.).

Lengyel, Béla. On the spectral theorem of selfadjoint operators. Acta Litt. Sci. Szeged 9, 174–186 (1939). [MF 609]

If R_z is the resolvent of a selfadjoint operator H in Hilbert space, it is well known that the function $w(z)=(R_zf, f)$, analytic above and below the real axis, has the integral representation

$$w(z)=\int_{-\infty}^{+\infty}\frac{d\rho(\lambda)}{\lambda-z}, \tag{1}$$

where $\rho(\lambda)$ is monotone-increasing and continuous on the right; and it is also well known that, at each point of continuity,

$$\rho(\lambda)=\lim_{y\to 0}\int_{-\infty}^{\lambda}\mathfrak{J}w(x+iy)dx, \tag{2}$$

where $z=x+iy$, $y>0$. The author justifies the use of (2) as a defining formula for $\rho(\lambda)$ and establishes (1) as a consequence, thus extending to non-bounded H a method employed by Hellinger in the bounded case [J. Reine Angew. Math. 136, 210–271 (1909)]. The relation $|z(R_zf, f)+(f, f)| \leqq K(f)/y$, which holds when f is in the domain of H, permits the successful application of function-theoretic methods here. It is well known that (1) then provides a means for setting up the spectral theory for H. *M. H. Stone.*

v. Neumann, J. On rings of operators. III. Ann. of Math. 41, 94–161 (1940). [MF 1008]

A factor is a ring of operators $\mathfrak{M}$ having the property that no element of it, except constant multiples of the identity, commute with every element of $\mathfrak{M}$. By means of a relative dimension function it has been established that five classes of factors I_n, I_∞, II_1, II_∞, III_∞ can be distinguished and that the first four classes are non-empty. [Cf. F. J. Murray and J. v. Neumann: On rings of operators, Ann. of Math. 37, 116–229 (1936).] In the present work, examples of class III_∞ are given. If A is in $\mathfrak{M}$, the rank of $\mathfrak{M}$ is the relative dimension of its range. If A is of finite rank, it is possible to define the trace of A. Those A's of $\mathfrak{M}$ which are of finite rank form a linear multiplicative set for which $T_r(AB^*)$ is an inner product. If we complete this

linear set with respect to the corresponding norm, we obtain what is called the Schmidt class. In case I_n, this set is $\mathfrak{M}$; in case I_∞, it corresponds to the operators of finite norm; in case II_1, it includes $\mathfrak{M}$, while in case III_∞, it consists simply of 0. Now if $E_1, E_2, \cdots, E_n$ are mutually orthogonal projections with $1=E_1+\cdots+E_n$, then $A_1=\sum_{i=1}^n E_iAE_i$ is of finite norm if A is. Each E_i in turn can be expressed as a sum of mutually orthogonal projections $E_i=E_{i,1}+\cdots+E_{i,r}$, and for this finer subdivision there is an $A_2=\sum E_{i,j}AE_{i,j}$. Continuing this process yields a sequence $A_1, A_2, \cdots$ with a limit (A); (A) is of finite norm if A is. Suppose that the E's are chosen so that, if F is a projection and not zero, (F) is not zero. Then if the Schmidt class is not simply 0, the set of (A)'s will contain non-zero elements of finite norm. If however no (A) is of finite norm, then $\mathfrak{M}$ must be a III_∞.

Next the examples of factors given in the previously cited joint paper are reexamined from another point of view. These examples are based on three elements, a set S, a measure function μ defined for subsets of S, and a group $\mathfrak{G}$ of transformations a of S, with certain properties one of which is that, if $S'\subset S$, then $\mu(aS')=\mu(S')$. In the present work, this condition is lightened to: $\mu(S')\neq 0$ implies $\mu(aS')\neq 0$. It is shown that certain of the resulting factors are such that there is no equivalent invariant measure and that these factors satisfy the condition given above for a III_∞. Two factors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to form a factorization if the ring generated by them is the full set of operators on the space and if their intersection is the set of multiples of the identity. It is shown here that factorizations occur involving a case II and a case III, thus completing the proof of the statement that, in a factorization, any combination of cases may occur except those in which only one factor is in a case I. *F. J. Murray* (New York, N. Y.).

Nakano, Hidegorô. Funktionen mehrerer hypermaximaler normaler Operatoren. Proc. Phys.-Math. Soc. Japan **21**, 713–728 (1939). [MF 1130]

The author defines and constructs measurable functions of a denumerable set of commutative normal operators in a Hilbert space. Such functions were originally treated by von Neumann. The present paper constitutes in part an extension of methods due to the reviewer; these had been applied to functions of one operator [Acta Litt. Sci. Szeged 7, 136–146 (1935)]. Let $N_1, N_2, \cdots$ be commutative normal operators and let $E(z_1), E(z_2), \cdots$ be their resolutions of the identity, each defined over a complex sphere S_i, $i=1, 2, \cdots$. The direct sum $S_1\times S_2\times\cdots$ is then the space in which measurable (with respect to $E_1(z_1), E_2(z_2), \cdots$) sets are defined. The expected theorems on measure are demonstrated, measurable functions are introduced, and ultimately the normal operator $\phi(N_1, N_2, \cdots)$ is constructed.

E. R. Lorch (New York, N. Y.).

Kodaira, Kunihiko. On some fundamental theorems in the theory of operators in Hilbert space. Proc. Imp. Acad., Tokyo **15**, 207–210 (1939). [MF 1139]

This paper presents new proofs of the well-known canonical decomposition ($A=WH$) and for normal operators, the integral representation theorem. In the latter result, the use of the integral representation for the W of a normal A is interesting, but as to the canonical decomposition, it is the reviewer's opinion that the method depends fundamentally on the graph as in von Neumann's original proof. The author neglects to refer to the paper of F. Riesz and E. R. Lorch [Trans. Amer. Math. Soc. **39**, 331–340 (1936)].

F. J. Murray (New York, N. Y.).

Steen, S. W. P. Introduction to the theory of operators. IV. Linear functionals. Proc. Cambridge Philos. Soc. **35**, 562–578 (1939). [MF 836]

In earlier papers [Proc. London Math. Soc. (2) **41**, 361–392 (1936); **43**, 529–543 (1937); **44**, 398–411 (1938)], the author has discussed two types of partially-ordered algebraic systems, respective instances of which are given by (1) the system of all self-adjoint operators (not necessarily bounded) "belonging to" an abelian operator-ring in Hilbert space, and (2) the system of all n-rowed Hermitian matrices over the complex field. In the present paper the author modifies his second set of postulates with a view to obtaining abstract analogues of certain parts of the theory of operator-rings developed by Murray and von Neumann [Ann. of Math. **37**, 116–229 (1936)]. The modification consists essentially in requiring that the linear space of "bounded" elements in the system be complete in a certain topology analogous to von Neumann's strong operator-topology. With this assumption it is possible to construct a dimension for idempotent elements and, more generally, a trace for general elements. It should be noted that a program similar in many respects to Steen's was inaugurated by von Neumann in Rec. Math. (Moscou), N.S. **1**, 415–482 (1936). *M. H. Stone* (Cambridge, Mass.).

Riesz, Frédéric. Sur quelques notions fondamentales dans la théorie générale des opérations linéaires. Ann. of Math. **41**, 174–206 (1940). [MF 1010]

This paper develops and applies certain ideas which the author formulated earlier [Atti del Congresso Internazionale dei Matematici, 1928, vol. 3, 143–148]. These ideas previously influenced H. Freudenthal [Nederl. Acad. Wetensch., Proc. **39**, 641–651 (1936)], whose results, as well as those of S. Kantorovitch, Garrett Birkhoff and others, are also extended and approached from a new angle. The investigation deals with partially ordered linear spaces, that is, with (additive) abelian groups in which a notion of non-negativity is defined. Denoting this by $f\geqq 0$, the partial ordering $f\geqq g$ or $g\leqq f$ is defined by $f-g\geqq 0$, and the usual properties are assumed. All other authors postulated further the lattice property, that is, the existence of a least upper bound and of a greatest lower bound for any two elements f, g. The author requires instead: Given any four elements $f_1, f_2, g_1, g_2\geqq 0$ with $f_1+f_2=g_1+g_2$, there exist four elements $h_{11}, h_{12}, h_{21}, h_{22}\geqq 0$ with $h_{11}+h_{12}=f_1$, $h_{21}+h_{22}=f_2$, $h_{11}+h_{21}=g_1$, $h_{12}+h_{22}=g_2$. This remarkable decomposition condition is weaker than the lattice postulate. The author then investigates those linear functions $L(f)$ in such a space, for which $L(f)$ is a real number not less than 0 when $f\geqq 0$. They and their differences form together a space satisfying the same conditions, the dual of the former. These dual spaces are shown to be lattices, and even to possess the stronger property of lattice completeness. The author establishes this, as well as a detailed spectral theory of the dual spaces, by very elegant direct methods and discusses numerous applications. These include various integrals of Stieltjes, Hellinger and Daniell. *J. von Neumann.*

Wintner, Aurel. On an ergodic analysis of the remainder term of mean motions. Proc. Nat. Acad. Sci. U. S. A. **26**, 126–129 (1940). [MF 1281]

The existence of a t-average for the argument of

$$S(t)=\sum_{j=1}^{n} a_j e(\lambda_j t+\varphi_j{}^0)$$

was previously proved [cf. H. Weyl, Amer. J. Math. **60**,

889–896 (1938)], so that the problem arises as to the behavior of $\omega(t)=\arg S(t)-\mu t$. The author proves that, except on a 0-set in the (φ_j^0)-space, the function $\omega'(t)$ is almost periodic (B). As a consequence the t-averages of $e^{-i\lambda t}\omega(t)$ exist for $\lambda\neq 0$. *E. R. van Kampen* (Baltimore, Md.).

Fukamiya, M. On dominated ergodic theorems in L_p ($p\geqq 1$). Tôhoku Math. J. **46**, 150–153 (1939). [MF 1177]

The author proves: (I) Let S be a set of finite measure and T a measurable, measure-preserving transformation of S into itself. If $f(P)\epsilon L_p(S)$, $p>1$, then the dominant

$$f^*(P)=\underset{0\leqq k<\infty}{\text{l.u.b.}}\frac{f(P)+\cdots+f(T^kP)}{k+1}$$

belongs to L_p, and

$$\int_S|f^*(P)|^p dV_P\leqq\left(\frac{p}{p-1}\right)^p\int_S|f(P)|^p dV_P.$$

(II) Under the same condition as in (I), if $f(P)\epsilon L_1(S)$, then a dominant $f^*(P)$ belonging to $L_{1-\epsilon}(\epsilon>0)$ exists, and

$$\int_S|f^*(P)|^{1-\epsilon}dV_P\leqq A\cdot\int_S|f(P)|dV_P+mS.$$

The second theorem, as the author recognizes, is contained in a stronger theorem since proved by Wiener [Duke Math. J. **5**, 1 (1939)]. *N. Wiener* (Cambridge, Mass.).

Yosida, Kôsaku. Asymptotic almost periodicities and ergodic theorems. Proc. Imp. Acad., Tokyo **15**, 255–259 (1939). [MF 1144]

The author calls a sequence of elements $X_1, X_2, \cdots$ in a complex Banach space B "asymptotically almost periodic" if to each positive ϵ there corresponds a relatively dense set S_ϵ of positive translation indices m such that

$$\overline{\lim_{n\to\infty}}\|X_{n+m}-X_n\|\leqq\epsilon.$$

He then connects this idea with ergodic theorems proved in former papers [Proc. Imp. Acad., Tokyo **14**, 286–291 and 292–294 (1938)], and at the same time strengthens these theorems and extends them to other topologies. Finally, let T be a one-to-one bounded linear transformation of B into B, and let the two-way sequence $\{T^n\cdot x\}$ be uniformly bounded and for each x strongly totally bounded. Then the author shows that T has at least one proper value of unit modulus, and he develops the Fourier analysis of T^n by means of the Bochner-von Neumann theory [Trans. Amer. Math. Soc. **37**, 21–50 (1935)]. *R. H. Cameron.*

Izumi, Shin-ichi. A non-homogeneous ergodic theorem. Proc. Imp. Acad., Tokyo **15**, 189–192 (1939). [MF 1136]

The object of this paper is to prove that the series $\sum_{n=1}^{\infty}(1/n)f(T^nP)$ converges almost everywhere under the following conditions: $f(P)$ belongs to L^2 over a space Ω of finite measure, and $\int_\Omega f(P)dP=0$. T is a measure-preserving transformation of Ω of "uniform mixture type," that is to say,

$$|T^nA\cdot B|=\frac{|A|\cdot|B|}{|\Omega|}+o\left(\frac{1}{\log^2|n|}\right)$$

holds uniformly for all measurable sets A, B such that $A\supset B$. The convergence almost everywhere of the above series is a stronger assertion than that of the means

$$M_n(T,P)=\frac{1}{n}\sum_{k=1}^{n}f(T^kP)$$

asserted by the ergodic theorem. Without the assumption $\int_\Omega f(P)dP=0$ it is shown that the series

$$\sum_{n=1}^{\infty}\frac{1}{n}[f(T^nP)-M_n(T,P)]$$

converges almost everywhere. *J. C. Oxtoby.*

Birkhoff, Garrett. An ergodic theorem for general semi-groups. Proc. Nat. Acad. Sci. U. S. A. **25**, 625–627 (1939). [MF 723]

Let G be a semi-group of linear operators on a finite or infinite dimensional Euclidean space, which do not increase distance. Then the means $\sum_a c_aT_a$ of the transforms T_ax of any point x converge to a fix-point y of the group. The convergence is in the following sense. For every neighborhood of y and every mean $x'=\sum_a c_aT_ax$ there is a mean $x''=\sum_a c_a'T_ax'$ such that all of the means $x'''=\sum_a c_a''T_ax$ lie in the given neighborhood of y. The theorem is unlike previous ergodic theorems in that it is no longer true if the assumption $|T_a|\leqq 1$ is replaced by $|T_a|\leqq M$ or if the Euclidean space is replaced by a uniformly convex one. The method of proof is that used by N. Wiener [Duke Math. J. **5**, 1–18 (1939)]. *N. Dunford.*

Alaoglu, L. and Birkhoff, G. General ergodic theorems. Proc. Nat. Acad. Sci. U. S. A. **25**, 628–630 (1939). [MF 724]

The following results concerning the theorem of the preceding review are stated. The theorem holds for a B-space E if and only if the closed convex hull of the transforms of every element of E contains one and only one fix-point of the group. If E is uniformly convex and G abelian then the theorem is true. The notion of convergence is equivalent to a notion of "convergence with respect to a sequence of nearly invariant measures." The assumption of the existence of a sequence of nearly invariant measures can be weakened to that of the existence of right and left invariant means defined over bounded real functions on G. In this case the theorem holds providing the closed convex hull of the transforms is weakly bi-compact. *N. Dunford.*

Theory of Probability

***Finetti, B. Compte rendu critique du colloque de Genève sur la théorie des probabilités.** Actual. Sci. Ind. 766. Conférences internationales de sciences mathématiques organisées à l'Université de Genève. Colloque consacré à la théorie des probabilités. VIII. Hermann & Cie, Paris, 1939. 65 pp.

Kamke, E. Zu meinem Aufsatz "Kritische Bemerkungen zu K. Marbe, Grundfragen der angewandten Wahrscheinlichkeitsrechnung und theoretischen Statistik." Jber. Deutsch. Math. Verein. **49**, 255–256 (1940). [MF 1218]

Concerning Jber. Deutsch. Math. Verein. **45**, 65–83 (1935).

Church, Alonzo. On the concept of a random sequence. Bull. Amer. Math. Soc. **46**, 130–135 (1940). [MF 1260]

This is an attempt to provide a definition of "random sequence" less stringent than that of von Mises and more stringent than that of admissible number or normal sequence. The fundamental notion is that of effective calculability as defined by the author by methods of symbolic logic [A. Church: An unsolvable problem of elementary number theory, Amer. J. Math. **58**, 345–363 (1936)]. The definition runs as follows: An infinite sequence $a_1, a_2, \cdots$ of 0's and 1's is a random sequence if the two following conditions are satisfied: (1) If $f(r)$ is the number of 1's among the first r terms of $a_1, a_2, \cdots$, then $f(r)/r$ approaches a limit p as r approaches infinity. (2) If ϕ is any effectively calculable function of positive integers, if $b_1=1$, $b_{n+1}=2b_n+a_n$, $c_n=\phi(b_n)$, and the integers n such that $c_n=1$ form in order of magnitude an infinite sequence $n_1, n_2, \cdots$, and if $g(r)$ is the number of 1's among the first r terms of $a_{n_1}, a_{n_2}, \cdots$, then $g(r)/r$ approaches the same limit p as r approaches infinity. The existence of such sequences is established by recourse to results of Doob and Wald, and the theory is thus logically satisfactory.

This paper seems intended as a contribution to the theory of sequences in connection with symbolic logic rather than as an attempt to bridge the gap between the theory of probability in collectives and the experimental applications. Thus the difficulty (among many others) of formulating with exactness the essential requirement that the value of p be unaltered if the subsequence be selected by an experimentally defined rule (such as deciding whether to accept or reject a_n by tossing a coin, by consulting a barometer, etc.) remains untouched. *B. O. Koopman.*

Larguier, Everett H., S.J. A matrix theory of n-dimensional measurement. Duke Math. J. **5**, 729–739 (1939). [MF 814]

Copeland uses the word variate to denote an infinite sequence of numbers having certain limit properties, and develops their theory by the methods of classical analysis. The present paper generalizes these ideas to what it calls n-dimensional variates, namely, infinite sequences of ordered sets of n numbers each. The notions of probability, independence, admissibility, etc., are defined in a manner similar to Copeland's by means of limits in the sequences, and many results fundamental in the sequence theory of probability are obtained. The paper avoids all controversial questions of epistemology. *B. O. Koopman.*

Lévy, Paul. Sur une loi de probabilité analogue à celle de Poisson et sur un sous-groupe important du groupe des lois indéfiniment divisibles. Bull. Sci. Math. **63**, 247–268 (1939). [MF 739]

The author studies in detail the distribution function whose density is given by the formula $f(x)=e^{-x}x^{t-1}/\Gamma(t)$ for $x\geqq 0$ and $f(x)=0$ for $x<0$ $(t>0)$. Many interesting points of the theory of infinitely divisible distributions are illustrated in this example. [For a summary of the results, cf. also C. R. Acad. Sci. Paris **207**, 1368–1370 (1938).] *M. Kac* (Ithaca, N. Y.).

Feldheim, Ervin. Sur quelques propriétés des lois de probabilité stables. Rev. Math. Union Interbalkan. **2**, 9–30 (1939). [MF 593]

Besides a general survey of the theory, the main purpose of the paper seems to be a proof that the density function $f(x)$ belonging to a stabile probability law cannot have more than one maximum; if $f(x)$ is an even function, it is monotone in $(0, \infty)$. *W. Feller* (Providence, R. I.).

Feller, Willy. Neuer Beweis für die Kolmogoroff-P. Lévysche Charakterisierung der unbeschränkt teilbaren Verteilungsfunktionen. Bull. Intern. Acad. Yougoslave. Cl. Sci. Math. Nat. **32**, 1–8 (1939). [MF 1148]

German version of a paper presented to the Academy in 1937. Meanwhile Khintchine published another proof [Bull. Univ. État Moscou, Sect. A. **1** (1938)]. *Autoreferat.*

Doeblin, W. Sur un problème de calcul des probabilités. C. R. Acad. Sci. Paris **209**, 742–743 (1939). [MF 840]

Let L be a law of probability (or distribution), $X_1, \cdots, X_n$ n independent variables, each distributed according to the law L, $S_n=X_1+\cdots+X_n$. The author states a necessary and sufficient condition for a set of probability laws to be the derived set $E'[L]$, that is, the set of all probability laws, each of which is the limit of a sequence of probability laws belonging to the quantities $a_k{}^2S_{n_k}-b_k$, where $\{n_k\}$ is an arbitrary increasing sequence of integers, and a_k, b_k are arbitrary constants. *P. Hartman* (Flushing, N. Y.).

Fukamiya, Masanori. The Lipschitz condition of random function. Tôhoku Math. J. **46**, 145–149 (1939). [MF 1176]

The author proves the theorem: If $\psi(t, \alpha)$ is Wiener's random function, then except for a set of values of α of measure zero

$$\overline{\lim_{\epsilon\to 0}}\, |\psi(t+\epsilon, \alpha)-\psi(t, \alpha)| \Big/ \left(\epsilon \log\left(\frac{\log 1/\epsilon}{\log 2}\right)\right)^{\frac{1}{2}}=(2\pi)^{\frac{1}{2}}$$

for almost all t in $(-\pi, \pi)$. The method of proof follows that of Wiener. The theorem is an extension of a theorem proved by Khintchine for each point in the line separately. *N. Wiener* (Cambridge, Mass.).

Ambrose, Warren. On measurable stochastic processes. Trans. Amer. Math. Soc. **47**, 66–79 (1940). [MF 1070]

This paper is a contribution to Doob's general theory of stochastic processes [Trans. Amer. Math. Soc. **42**, 107–140 (1937)]. Let Ω be the space of all real functions of t, $-\infty<t<\infty$, and let $P(M)$ be a probability measure defined in the well-known manner on the Borel field of sets determined by the collection of so-called cylindric sets. Doob's theory rests on the idea of considering, instead of Ω, a subset Ω' whose outer P-measure is 1; if M' is the intersection of Ω' with a P-measurable set M, we put $P'(M')=1$; the space Ω' together with the so-defined probability measure $P'(M')$ is called by Doob a stochastic process. If ω is a point of Ω', that is to say, some function $g(t)$, denote by $x_s(\omega)$ the value $g(s)$. Thus $x_s(\omega)$ is a function defined on the product space of the t-axis T with Ω'. We introduce in $T\times\Omega'$ a measure by means of the product of the Lebesgue-measure on T with $P'(M')$. If, then, the function $x_s(\omega)$ is measurable on $T\times\Omega'$, the process Ω' with $P'(M')$ is called measurable. The main object of the present paper is to find necessary and sufficient conditions that the space Ω with a measure $P(M)$ contains a measurable stochastic process Ω' with the corresponding measure $P'(M')$. Such a condition is the existence of a measurable function on $T\times\Omega$ which for each fixed t equals $x_t(\omega)$ at almost all ω-points. From this, different necessary and sufficient conditions are derived, in particular, that for almost all t and all $\epsilon>0$, $P(|x(t+h)-x(t)|>\epsilon)\to 0$ as $h\to 0$ on a h-set which may depend on t but

not on ϵ, and having density 1 at $h=0$. This condition is due to Kolmogoroff, but has so far been communicated, without proof, only in a footnote by Doob [Duke Math. J. **4**, 752–774 (1938)]. Several other conditions are in terms of conditional probabilities. Finally the author restates and gives a new proof of a result of Doeblin [Bull. Math. Soc. Roum. Sci. **39**, 57–115]: A process is called a Markoff process if, with Doob's notation for the conditional probabilities, $P(x_{t_1}(\omega), \cdots, x_{t_n}(\omega); M) = P(x_{t_n}(\omega); M)$ whenever $t_1 < t_2 < t_3 < \cdots < t_n$. (Kolmogoroff uses the term "stochastically definite" for such processes.) Suppose, moreover, that for every $\epsilon > 0$ we have $P(x_t(\omega); |x_t(\omega) - x_{t+h}(\omega)| < \epsilon) \to 1$ as $h \to 0$ uniformly in ω and t. Then Ω contains a measurable stochastic process whose elements are continuous almost everywhere on the t-axis with probability 1, and each ω is continuous at $t = t_0$ with probability 1 for every t_0.

W. Feller (Providence, R. I.).

Lévy, Paul. Sur certains processus stochastiques homogènes. Compositio Math. **7**, 283–339 (1939). [MF 799]

The author continues, and gives the details of researches announced in, the Paris Comptes Rendus [**207**, 1368–1370 (1938); **208**, 318–321 (1939)]. These concern the character of a random function $X(t)$ for which, if $t_1 < \cdots < t_n$, $\{X(t_j) - X(t_{j-1})\}$ form an independent set of chance variables whose distributions are unaffected by translations of the t-axis. A few of the theorems obtained go back to Bachelier [cf. references to Bachelier's early work in his book: Les nouvelles méthodes du calcul des probabilités, Paris, 1939]. The following are characteristic of the many results. They will be stated under the assumption that the distribution of $X(t+h) - X(t)$ is Gaussian; other cases are, however, also treated in the paper. The distribution of $\max_{0 \le t' \le t} [X(t') - X(0)]$ is found. If $T(y)$ is the first t-value greater than 0, where $[X(t) - X(0)] = y$, the distribution of the chance variable $T(y)$ is found. Let E be the t-set where $X(t) = X(0)$. It is shown that $mE = 0$ with probability 1. The sum of the lengths of those t-intervals of the complement of E which are in $0 \leqq t \leqq u$ and which have length not greater than l is an infinitesimal ($l \to 0$) equivalent to $sl^{\frac{1}{2}}$, where s is a chance variable which is shown to have important properties. For example, the distribution function of the measure of the t-set in the interval $(0, u)$ for which $X(t) < x$ is found, and the derivative of this function with respect to t, at $x = 0$, is shown to be $s(\pi/2)^{\frac{1}{2}}$, with probability 1.

J. L. Doob (Urbana, Ill.).

Cramér, Harald. On the theory of stationary random processes. Ann. of Math. **41**, 215–230 (1940). [MF 1012]

An n-dimensional random process is discussed, that is, an n-dimensional random variable $Z(t) = \{X_1(t), \cdots, X_n(t)\}$ is considered, depending on the parameter t, $-\infty < t < \infty$. It is supposed as usual that the probability that $Z(t)$ satisfy certain conditions of an elementary type is given. These probabilities determine the probability relations of the process. The $X_j(t)$ are complex-valued. The process is stationary and stationarily correlated: Expectation $\{X_\mu(t)\}$ is independent of t, and Expectation $\{X_\mu(t)\bar{X}_\nu(u)\} = R_{\mu\nu}(t-u)$ depends only on $t-u$. It is further assumed that Expectation $X_\mu(t) = 0$, and that no $X_\mu(t)$ is identically a constant (with probability 1), to avoid unessential complications. The regularity hypothesis that $R_{\mu\mu}(t)$ is continuous at 0 is made, and this is shown to imply the continuity of all the correlation functions $R_{\mu\nu}(t)$ for all t. Conditions on the $R_{\mu\nu}(t)$ are found which are necessary and sufficient that they are the correlation functions of such a process. [The results include conditions found by Khintchine, Math. Ann. **109**, 604–615 (1934) in the one-dimensional case]. The $R_{\mu\nu}(t)$ must be given by Fourier integrals: $R_{\mu\nu}(t) = \int_{-\infty}^{\infty} e^{itx} dF_{\mu\nu}(x)$, where the $F_{\mu\nu}$ are of bounded variation and continuous on the right, and are such that the quadratic form $\sum_{\mu,\nu=1}^{n} z_\mu \bar{z}_\nu [F_{\mu\nu}(\beta) - F_{\mu\nu}(\alpha)]$ is non-negative definite for any $\beta \geqq \alpha$. If the correlation functions satisfy these conditions, there is even an n-dimensional process whose X_j are normally distributed. The real case (real X_j) and the discrete process (t running through the integers only) are also treated.

J. L. Doob.

Kakutani, Shizuo. Some results in the operator-theoretical treatment of the Markoff process. Proc. Imp. Acad., Tokyo **15**, 260–264 (1939). [MF 1145]

Let $P^{(n)}(t, E)$ be the probability that a point t is transferred in n steps of a Markoff process into a set E. Let T be the operator taking the set function $x(E)$ into $y(E) = \int P^{(1)}(t, E)x(dt)$. The transformation T is a linear operator on the Banach space $\mathfrak{M}$ of set functions $x(E)$ (norm of $x(E)$ = total variation of $x(E)$), and is supposed to satisfy the condition of Kryloff and Bogolioùboff [C. R. Acad. Sci. Paris **204**, 1386–1388, 1454–1455 (1937)]: there is an integer m and a completely continuous operator V mapping $\mathfrak{M}$ on itself, such that $\|T^m - V\| < 1$. The author continues the work of Yosida [Proc. Imp. Acad., Tokyo **14**, 363–367 (1938)] in analyzing the asymptotic behavior of $P^{(n)}(t, E)$ ($n \to \infty$) by means of the iterates of T, getting somewhat more powerful and detailed results than those of Kryloff and Bogolioùboff, Fréchet, Doob, Doeblin, on the convergence of

$$\frac{1}{N} \sum_{1}^{N} P^{(n)}(t, E), \qquad \text{as } N \to \infty,$$

the division of t-space into ergodic and dissipative parts, etc.

J. L. Doob (Urbana, Ill.).

Silberstein, Ludwik. On a hystero-differential equation arising in a probability problem. Philos. Mag. **29**, 75–84 (1940). [MF 1160]

Consider random-events subjected to Poisson's distribution, the probability of the occurrence of exactly n events during any time-interval of length t being $t^n e^{-t}/n!$. The author is concerned with the probability $P(t)$ that a randomly chosen interval of length t contains at least two events such that the time elapsed between them does not exceed some given constant τ. Obviously $P(t) = 1 - e^{-t}(1+t)$ for $0 < t < \tau$, and it is easily verified that $P'(t) = 1 - P(t) - [1 - P(t-\tau)]e^{-\tau}$ for $t > \tau$; this is the equation to which the title refers. By the substitution $1 - P(t) = e^{-t}f(t)$, it reduces to $f'(t) = f(t-\tau)$, which determines uniquely $f(t)$ for $t > \tau$ if the values for $0 < t < \tau$ are known; the explicit solution in the interval $n\tau < t < (n+1)\tau$ depends on n integrations. In addition the author considers a treatment of $f'(t) = f(t-\tau)$ analogous to that of linear differential equations by putting $f(t) = \sum a_k e^{m_k t}$, where m_k are the (infinitely many) roots of $me^{m\tau} = 1$. The problem is said to arise in the quantum-theory of photographic reciprocity failure.

W. Feller.

Ruark, Arthur. The time distribution of so-called random events. Phys. Rev. **56**, 1165–1167 (1939). [MF 699]

Consider events occurring according to Poisson's probability distribution, the probability of the occurrence of exactly n events during the time t being $(ft)^n e^{-ft}/n!$, $f = \text{const.} > 0$. The author aims at the calculation of the con-

ditional probability $G(T; n, D)$ that the interval between consecutive events, chosen at random in $(0, D)$, lies between T and D when it is known that during $(0, D)$ there had been $n \geqq 2$ events. The author's formula is, however, incorrect since, for $T=0$, any positive value may result (instead of 1).

W. Feller (Providence, R. I.).

Theoretical Statistics

★Jeffreys, Harold. Theory of Probability. Oxford University Press, Oxford, 1939. vi+380 pp. $7.00.

This work is a discussion of the theory of statistical inference, and of the foundations of probability, not a discussion of the probability calculus. Proofs are usually only indicated. Contents: I. Fundamental notions, including a postulational approach to statistical inference. The probability of a proposition is thought of as the reasonable degree of confidence one has in it. It is postulated that, given some set of data, these degrees of confidence form an ordered series of ordinal type less than or equal to that of the continuum $0 \leqq x \leqq 1$, and that if (given p) (i) q, q' cannot both be true, and (ii) r, r' cannot both be true, then the relations Prob. (q) = Prob. (r), Prob. (q') = Prob. (r') imply that Prob. $(q$ or $q')$ = Prob. $(r$ or $r')$. As a convention, numbers are assigned to these probabilities in such a way that probability becomes additive in the usual sense. It is not clear whether the axioms are to furnish an abstract axiomatization of inductive logic, including a formalization of the word "reasonable" in the phrase "reasonable degree of confidence," or whether they are merely to furnish a set of practical rules of procedure. This vagueness makes somewhat doubtful the admissibility of the above convention, whose relation to the axioms is not fully discussed. A further convention is usually made that an isolated proposition about whose probability nothing is known is to have probability $\frac{1}{2}$. II. Direct probabilities: a discussion of elementary probability theorems, the Poisson law, the normal law, Pearson's distributions, etc. III. Estimation problems, including a discussion of least squares, correlation, etc. The standpoint adopted goes back to Bayes; the unknown parameters are supposed to have given distributions. IV. Approximate methods and simplifications: maximum likelihood, efficiency of estimates, effects of grouping, rank correlation, etc. V, VI. Significance tests: a careful discussion of many critical statistical problems, with many specific examples. VII, VIII. Frequency definitions and direct methods, general questions: a discussion of various definitions of probability and points of view in statistics. The author usually does not distinguish between the probability calculus and the interpretation problem. Adherents of the points of view not adopted by the author might well object to his descriptions of these points of view and of their purpose. Thus the treatment of the probability calculus which identifies probabilities with measures of point sets is rather irrelevantly scorned [p. 304] because in spite of "the elaborate use of ϵ, $o(n^{-\frac{1}{2}})$, and 'almost everywhere,'" it is not always made clear how the measures are determined in practical problems. As the author says, however, many statisticians, who do not agree in their definitions, agree in their practical applications and results. In this spirit, then, whatever the disagreement with the underlying concepts, all statisticians can welcome a book containing so much solid and detailed discussion of critical statistical problems.

J. L. Doob (Urbana, Ill.).

★Croxton, Frederick E. and Cowden, Dudley J. Applied General Statistics. Prentice-Hall, Inc., New York, 1939. xviii+944+xiii pp. $4.00.

In good accord with the title, this is not a mathematical book, but a compendium of methods and of a number of useful tables. The use of both is explained in a number of examples. The contents of particular chapters are: Introduction. Statistical data. Statistical tables. Graphic presentation. Ratios and percentages. Frequency distributions. Measures of central tendency. Dispersion, skewness and curtosis. Reliability and significance of statistical measures. Time series. Trend. Periodic movements. Seasonal and cyclical movements. Index numbers. Correlation: linear, nonlinear, multiple and partial. Correlation of time series and forecasting. The authors obviously endeavored to make the book intelligible to a reader with the least mathematical background. It seems that they have somewhat exaggerated.

J. Neyman (Berkeley, Calif.).

★Tschuprow, A. A. Principles of the Mathematical Theory of Correlation. William Hodge and Company, Ltd., London, 1939. x+194 pp. 12/6. (Nordemann Publishing Co., Inc., New York. $3.75.)

This is a translation (by M. Kantorovitsch) of the original German edition of 1925, which became a standard work by its treatment of the systematic error (or bias) committed when the empirical correlation coefficient (or related quantities) are used as estimates of the corresponding theoretical quantities. The theory of estimation and of testing the significance of observed differences between empirical observations has made important progress since the first appearance of this book, but the modern statistician will still read these beginnings of the modern theory with historical interest. A large portion of the book is devoted to a discussion of general principles underlying the correlation and its applications. In approach and outlook the author is admittedly closely following the late Karl Pearson.

W. Feller.

Masuamya, Motosaburô. Correlation between tensor quantities. Proc. Phys.-Math. Soc. Japan 21, 638–647 (1939). [MF 1128]

The purpose of this paper is to give satisfactory extensions of the ordinary definition of correlation coefficient (C.C.) to vector and tensor sets and fields; actually only vectors are treated. The space is Euclidean 3-space. Consider two vector sets; let $\underset{\alpha}{\mathbf{a}}$, $\underset{\alpha}{\mathbf{b}}$ $(\alpha=1, 2, \cdots, N)$ be the deviations from the respective means. Three C.C.'s are defined. (I) The primitive C.C.

$$r=(\mathbf{a}, \mathbf{b})/\{(\mathbf{a}, \mathbf{a})^{\frac{1}{2}}(\mathbf{b}, \mathbf{b})^{\frac{1}{2}}\}, \quad (\mathbf{a}, \mathbf{b})=\sum_{\alpha=1}^{N} \underset{\alpha}{\mathbf{a}} \cdot \underset{\alpha}{\mathbf{b}}.$$

This is a very strong definition, since $r=1$ implies $\underset{\alpha}{\mathbf{a}}=\lambda \underset{\alpha}{\mathbf{b}}$, where λ is some scalar. (II) The modified C.C. (unsymmetric)

$$r^*=(\mathbf{a}, \mathbf{a}^*)/\{(\mathbf{a}, \mathbf{a})^{\frac{1}{2}}(\mathbf{a}^*, \mathbf{a}^*)^{\frac{1}{2}}\},$$

where $\underset{\alpha}{\mathbf{a}}^*$ is that vector set of the form $T \cdot \underset{\alpha}{\mathbf{b}}$ (T some tensor) which minimizes $(\mathbf{a}-\mathbf{a}^*, \mathbf{a}-\mathbf{a}^*)$; $r^*=1$ implies $\underset{\alpha}{\mathbf{a}}=T \cdot \underset{\alpha}{\mathbf{b}}$. (III) The symmetric C.C.

$$R=|[a_i b_j]|/\{|[a_i a_j]|^{\frac{1}{2}}|[b_i b_j]|^{\frac{1}{2}}\}, \quad [a_i b_j]=\sum_{\alpha=1}^{N} \underset{\alpha}{a_i} \underset{\alpha}{b_j},$$

where $|\ |$ indicates determinant and the subscripts i, j the components of the vectors; $R=1$ implies $\underset{\alpha}{\mathbf{a}}=T \cdot \underset{\alpha}{\mathbf{b}}$.

J. L. Synge (Toronto, Ont.).

Derksen, J. B. D. On some infinite series introduced by Tschuprow. Ann. Math. Statistics **10**, 380–383 (1939). [MF 759]

In order to get approximate values for the moment characteristics in samples of N of quotients of moment characteristics, such as the coefficient of correlation, Tschuprow and others after him have expanded the denominators by the binomial series and taken the mathematical expectations of the separate terms, obtaining a series in descending powers of N. But such series are generally divergent if the expectations are taken over the full range of the variables entering into the sample, whereas, if the range is restricted, they are convergent. In this note the author applies the results Slutsky obtained for conditionally aleatory variables [Über stochastische Asymptoten und Grenzwerte, Metron **5**, no. 3, 79 (1925)] to show that if the probability that the denominator differs from its expected value by more than a fixed amount converges stochastically to 0 with increasing N, then such expressions as Tschuprow obtained give valid approximations for N sufficiently large. *C. C. Craig.*

Starkey, D. M. The distribution of the multiple correlation coefficient in periodogram analysis. Ann. Math. Statistics **10**, 327–336 (1939). [MF 754]

Hotelling [Amer. J. Math. **61**, 440–460 (1939)] considered in the periodogram analysis (*) $Y=a+b\cos(kx+\epsilon)$, as leading under the least square hypothesis to "the correlation R between the observed values y and the values Y computed from the regression equation (*)." Starkey calls this R the "multiple correlation between the observed and fitted values." This use of "multiple" should be distinguished from that which implies the use of more than two sets of related variates. Defining θ as $\cos^{-1} R$, and assuming that θ is not very close to $\pi/2$, and that n is large, the author obtains, after a rather detailed analysis involving definite integrals taken between limits that are functions of n or of $n^{1/2}$, "the approximate frequency function for the quantity θ in the case of periodogram analysis" $(\pi/6)^{1/2}n^{5/2}\theta^{n-5}d\theta$. The author points out that this investigation does not cover adequately some important cases, such as a test for the null hypothesis. *E. L. Dodd* (Austin, Tex.).

Eisenhart, C. A note on a priori information. Ann. Math. Statistics **10**, 390–393 (1939). [MF 762]

Gumbel, E. J. La probabilité des hypothèses. C. R. Acad. Sci. Paris **209**, 645–647 (1939). [MF 708]

Let x be a random variable, n values of which can be given by independent observations, and $p(x)$ its elementary probability law completely specified by a hypothesis h. As it is known, the hypothesis h is equivalent to the statement that the variable $y=\int_{-\infty}^{x}p(x)dx$ follows a rectangular distribution between zero and unity. Let x_m be the mth of the observed x's in the order of their magnitude and y_m be the corresponding value of y. Then the expected value of y_m is $\bar{y}_m=m/(n+1)$ and the author suggests a method of testing h based on the comparison of the observed values of y_m with their expectations. *J. Neyman* (Berkeley, Calif.).

Wald, Abraham. Contributions to the theory of statistical estimation and testing hypotheses. Ann. Math. Statistics **10**, 299–326 (1939). [MF 753]

This paper deals with ideas generalizing many conceptions basic to the theories of testing statistical hypotheses and of estimation. Consider a set X of n observable random variables, the probability law of which is known except for the value of k parameters, the set of which will be denoted by a single letter θ and represented by a point in the k dimensional space Ω. The letter w will denote a subset of Ω and the assumption that $\theta\epsilon w$ will be a statistical hypothesis, simple if w contains but one point and composite otherwise. The author is concerned with the case where the nature of an applicational problem leads to the consideration of a set S of subsets w which may be overlapping or not. The observed values of X will determine a sample point E, the position of which in the sample space M will lead to the acceptance of one or another out of the alternative hypotheses, equivalent to the assumption of θ being an element of some particular $w\epsilon S$. It follows that, apart from considering the set S of subsets of Ω, we must consider a partitioning of the sample space M into a system Σ of nonoverlapping sets M_w, described as regions of acceptance, being in one to one correspondence with subsets w. If then the observations determine $E\epsilon M_w$, we shall state that $\theta\epsilon w$. Evidently, if S contains but two subsets w_1 and $w_2=\Omega-w_1$, we have the usual problem of testing a statistical hypothesis. On the other hand, if the regions of acceptance M_w are selected in accordance with certain restrictions, then the problem of the author becomes that of estimation with the subsets w playing the role of "estimating sets" [J. Neyman, L'estimation statistique traitée comme un problème classique de probabilité, Actual. Sci. Ind. no. 739 (1938)]. Out of the many conceptions introduced, the following seem to play the most important role. Consider a system S of subsets w and some system Σ of regions of acceptance M_w. Let $W(\theta, w)$ represent the weight of an error which would be committed by assuming that the parameter point belongs to w, while its true position is θ. For values $\theta'\epsilon w$, the weight $W(\theta', w)=0$. The choice of $W(\theta, w)$ does not constitute any statistical problem and must be determined in accordance with the practical considerations; for example, $W(\theta, w)$ may represent the financial loss caused by the error. Now the subset w may be considered as a function $w(E)$ of the sample point E. Namely, if $E\epsilon M_{w'}$, then $w(E)=w'$. With this notation $W(\theta, w(E))$ will also be a function of E and thus a random variable, representing the weight of an error to be committed if the observations determine the point E while the true position of the parameter point is θ. Let $p(E|\theta)$ be the elementary probability law of X, depending on θ. The integral

$$r\{\theta|\Sigma, W(\theta, w)\}=\int_M W(\theta, w(E))p(E|\theta)dE,$$

representing the expectation of the financial loss to be incurred by committing errors, is considered as a function of θ, and described as the risk function associated with the system Σ of regions of acceptance and relative to the weight $W(\theta, w)$. Starting with the suggestion that the smaller, in general, the risk function, the better the system of regions of acceptance, the author arrives at various conceptions of best estimates, etc., and proves many interesting properties of the same. *J. Neyman* (Berkeley, Calif.).

Kullback, Solomon. A note on Neyman's theory of statistical estimation. Ann. Math. Statistics **10**, 388–390 (1939). [MF 761]

For purposes of statistical estimation Neyman [e.g., Philos. Trans. Roy. Soc. London, Ser. A **236** (1937)] uses functions $\underline{\theta}(E)$ and $\bar{\theta}(E)$ for which, in his notation, $P\{\underline{\theta}(E)L\theta_1L\bar{\theta}(E)|E\}\equiv\alpha$. The empirical meaning of this equation is interpreted by Neyman by the law of large numbers. The purpose of the present paper is to elucidate the point by another argument. (It seems, however, to the

reviewer that it is essentially Neyman's reasoning put into a formula.) *W. Feller* (Providence, R. I.).

Yates, F. An apparent inconsistency arising from tests of significance based on fiducial distributions of unknown parameters. Proc. Cambridge Philos. Soc. **35**, 579–609 (1939). [MF 837]

The first part of this paper deals with the notions of fiducial argument which have been somewhat more precisely set forth in papers by other authors. The remainder of the paper is devoted to the Behrens-Fisher test of the significance of the difference of means of pairs of samples independently drawn from normal populations in which the variances are not assumed to be equal. Some indication is given as to why this test yields a smaller percentage of significant differences than would be the case for a t-test involving a pooled estimate of the variance for populations with a fixed variance ratio. *S. S. Wilks.*

Fisher, R. A. A note on fiducial inference. Ann. Math. Statistics **10**, 383–388 (1939). [MF 760]

On the occasion of some remarks by Bartlett [Ann. Math. Statistics **10**, 129–138 (1939)] the author states that, and why, he rejects Bartlett's test of significance for the difference between means of two samples from normal populations. The only test appropriate to the problem is, in the author's opinion, that given by Behrens. [Cf. papers by the author, Ann. Eugenics **6**, 91–98 (1935) and **8**, 370–375 (1937).] *W. Feller* (Providence, R. I.).

Lengyel, B. A. On testing the hypothesis that two samples have been drawn from a common normal population. Ann. Math. Statistics **10**, 365–375 (1939). [MF 757]

The problem considered in this paper is that of testing the hypothesis that two samples of 2, 3 and 4 variables and of equal size have been drawn from identical normal populations. The criterion used for making the test is the Nth root of the likelihood ratio, where N is the sample size. The distribution functions of this criterion for the several cases are approximated by fitting Pearson Type I distribution functions, using two moments. An investigation is made of the degree of approximation for large samples, but no such study is made for small samples. One and five percent significance levels of the criterion are obtained from the fitted functions for samples of sizes 10 to 50. An illustrative example based on dementia praecox data is included.
S. S. Wilks (Princeton, N. J.).

Newman, D. The distribution of range in samples from a normal population, expressed in terms of an independent estimate of standard deviation. Biometrika **31**, 20–30 (1939). [MF 258]

Tippett (in 1925) prepared tables of the mean range within samples all taken from a normal population in terms of the population standard deviation. This step has proved the incentive for a considerable amount of computational work with the object of making use of the range of a sample, rather than its sum of squares, for rapidly estimating σ. Following a suggestion of "Student," a study is here made of the sampling distribution of $q=w/s$, where w is the range in a sample of n observations from a normal population with (unknown) standard deviation σ, and s^2 is an independent and unbiased estimate of σ^2 based on f degrees of freedom. Basic frame-work values are computed, and, by interpolation, tables for the 5% and 1% points in terms of 12 values of n, and 21 values of f, are here obtained and published. These tables rest to some extent on an empirical basis. Instances are worked out illustrating their ease of application. *A. A. Bennett* (Providence, R. I.).

Welch, B. L. On the distribution of maximum likelihood estimates. Biometrika **31**, 187–190 (1939). [MF 261]

Let x be a random variable and $p(x|\theta)=\exp(\phi(x,\theta))$ be its elementary probability law, depending on a parameter θ, the value of which is unknown. Let further $x_1, x_2, \cdots, x_n$ denote n independent observations of x. The maximum likelihood estimate T of θ is then one of the roots of the equation $\sum\phi'(x_i, T)=0$, where ϕ' denotes the derivative with respect to θ. The author considers the problem of an approximate calculation of the probability that $T>T_0$, where T_0 is any fixed number. He notices that frequently, for example, if $\sum\phi''(x_i, T_0)$ is essentially negative, the inequality $T>T_0$ is equivalent to $\sum\phi'(x_i, T_0)>0$. Consequently, if θ is fixed, the probability of $T>T_0$ will be identical with that of $\sum\phi'(x_i, T_0)>0$. The latter can frequently be easily approximated by first calculating the moments of $\phi'(x, T_0)$, then the cumulants of $y=\sum\phi'(x_i, T_0)$, and finally by fitting a Pearson curve or using any other expansion. Once the distribution $p(y|T_0, \theta)$ of y is sufficiently approximated, the required probability will be found by integrating $p(y|T_0, \theta)$ between zero and infinity.
J. Neyman (Berkeley, Calif.).

Laderman, Jack. The distribution of "student's" ratio for samples of two items drawn from non-normal universes. Ann. Math. Statistics **10**, 376–379 (1939). [MF 758]

A formal expression is derived for the distribution of the "Student" ratio $t=\bar{x}/s$ for samples of two drawn from any continuous universe. *W. A. Shewhart.*

Johnson, N. L. and Welch, B. L. On the calculation of the cumulants of the χ-distribution. Biometrika **31**, 216–218 (1939). [MF 264]

The cumulants of the Chi-distribution with f degrees of freedom are here studied with the aim of providing considerable accuracy by methods less laborious than calculating moments about zero, and then correcting these in the usual way. From polynomial formulas for the first six cumulants in terms of the first moment about zero, a table to six decimal figures for the value of the first six cumulants for f from 1 to 8 inclusive and a second adapted to larger values of f are computed. *A. A. Bennett.*

David, F. N. On Neyman's "smooth" test for goodness of fit. I. Distribution of the criterion ψ^2 when the hypothesis tested is true. Biometrika **31**, 191–199 (1939). [MF 262]

For large samples Neyman [Skand. Aktuarietidskr. **20**, 149–199 (1937)] showed that his criterion ψ^2 of smoothness of fit of order k is distributed as χ^2 with k degrees of freedom. The present paper is mainly concerned with throwing more light on what constitutes a large sample in this connection. The first four moments of the distribution of ψ^2 in samples for $k=1$ and 2 are calculated and Pearson curves are fitted to them, the probability levels resulting for different sample sizes being studied. The conclusion is that for samples of 20 or over no great error is made in assuming them large within the meaning above for these values of k. There is some further critical discussion of the usefulness of the test with mention of further research that should be undertaken relating to this question. *C. C. Craig.*

Welch, B. L. Note on discriminant functions. Biometrika **31**, 218–220 (1939). [MF 265]

Suppose that a population consists of two specified groups Π_1 and Π_2, and that each individual with q measured characters $x_1, \cdots, x_q$ is to be placed in either Π_1 or Π_2. The probability distributions $p_i(x_1 \cdots x_q)$ in Π_i are supposed known. A classification rule is required such that the probability of a misclassification be the same for individuals of both groups and as small as possible. This rule is shown to consist in assigning any individual to Π_1 if and only if its measured characters belong to the region R defined by $p_1/p_2>k$, where k is a constant determined by $\int_R(p_1+p_2)d\tau=1$. The function p_1/p_2 is called the "discriminant function." *W. Feller* (Providence, R. I.).

Bishop, Morris C. A note on computation for analysis of variance. Ann. Math. Statistics **10**, 393–399 (1939). [MF 763]

The author gives a simple computational procedure embodying checks for obtaining the constituent items used in the analysis of variance for randomized blocks, Latin squares and other balanced arrangements. *S. S. Wilks* (Princeton, N. J.).

Jeffreys, Harold. Random and systematic arrangements. Biometrika **31**, 1–8 (1939). [MF 255]

An attempt to reconcile the conflicting views in the controversy, originated by R. A. Fisher and "Student," concerning the merits and the demerits of random and systematically arranged experiments. *J. Neyman.*

McCarthy, M. D. On the application of the z-test to randomized blocks. Ann. Math. Statistics **10**, 337–359 (1939). [MF 755]

Consider an experiment in which s varieties are compared in n randomized blocks. Denote by $x_{ij(k)}$ the yield to be obtained from the jth plot in the ith block if it be sown with the kth variety; $x_{ij(k)}$ is considered as a random variable and its expectation denoted by $X_{ij(k)}$. The difference $x_{ij(k)}-X_{ij(k)}=\epsilon_{ij(k)}$ depending on inaccuracies in cultivation, etc., is described as "experimental error" in $x_{ij(k)}$. Symbols $X_{i\cdot(k)}$ and $X_{\cdot\cdot(k)}$ denote the means of the numbers $X_{ij(k)}$ taken over the ith block and over the whole experimental field, respectively. The difference $X_{ij(k)}-X_{i\cdot(k)}=\eta_{ij(k)}$ is described as the "soil error" in $x_{ij(k)}$ and $X_{i\cdot(k)}-X_{\cdot\cdot(k)}=B_i$, supposed to be independent of k, as the contribution of the ith block. With this notation $x_{ij(k)}=X_{\cdot\cdot(k)}+B_i+\eta_{ij(k)}+\epsilon_{ij(k)}$ and the purpose of the experiment is understood to consist in the comparison of the numbers $X_{\cdot\cdot(k)}$. If the blocks are randomized so that the plots within each block are assigned to particular varieties at random, then, before this is done, the value of j is not determined and we may write $x_{i(k)}=X_{\cdot\cdot(k)}+B_i+\eta_{i(k)}+\epsilon$ to denote the random variable representing the yield of the kth variety to be obtained from the plot in the ith block, which will be selected at random for this particular variety. Here $\eta_{i(k)}$ is a random variable with a finite number s of possible values $\eta_{ij(k)}$, for $j=1, 2, \cdots, s$. It follows that, if $k\neq l$, then $\eta_{i(k)}$ and $\eta_{i(l)}$ are of necessity correlated, since, if $\eta_{i(k)}$ assumes the value $\eta_{ij(k)}$, then $\eta_{i(l)}$ may have any value $\eta_{it(l)}$, for $t=1, 2, \cdots, s$, except for $\eta_{ij(l)}$. The nature of technical errors is largely unknown and the author assumes that they are mutually independent and independent from the η's. As a result of the above analysis and hypothesis concerning ϵ, the variables $x_{i(k)}$ and $x_{i(l)}$ appear to be correlated. As the traditional theory of randomized blocks ignores such correlations, the purpose of the paper reviewed is to evaluate the effect of this oversight on the results of application of the usual z-test. While keeping the traditional hypothesis that the x's are normally distributed, in order to obtain a theory closer to actual situation, the author assumes the existence of such correlations between the x's as would result from the connection between the η's, and deduces the probability law of z. The numerical values of significance limits are given only for $s=3$ and appear to be quite sharply different from those determined by the traditional theory. In general, the greater the value of σ_η as compared to σ_ϵ, the greater is the inadequacy of the traditional theory, which tends to overestimate the significance.

J. Neyman (Berkeley, Calif.).

Applications of the Theory of Probability, Economics

Hadwiger, H. Über die Integralgleichung der Bevölkerungstheorie. Mitt. Verein. Schweiz. Versich.-Math. **38**, 1–14 (1939). [MF 303]

The author solves the equation

$$G(t)=\int_0^a G(t-\xi)K(\xi)d\xi,$$

with $0<a\leqq\infty$, by using the Laplace transforms. Under some regularity conditions the values of $G(t)$ for $-a<t<0$ can be arbitrarily prescribed, and the solution is then uniquely determined for $t>0$. The equation has previously been treated, especially by Lotka [most recently in Skand. Aktuarietidskr. **1933**], in connection with investigations in industrial replacement and population growth. In the latter theory $G(t)$ equals the female birthrate, $K(\xi)$ is the compound probability that a newborn female live at least ξ years and have a daughter at the age ξ; finally a is the maximal age of fertility. *W. Feller* (Providence, R. I.).

Kriehl, Lothar. Das Problem der Rassenvermischung und seine Bedeutung für die dynamische Sterbetafel. Arch. Math. Wirtsch.-Sozialforschg. **5**, 166–188 (1939). [MF 719]

The author studies a population consisting of two races and their intermixture. Six different intensities $\mu_{i,k}$ of nativity are introduced according to the races of the parents; it is assumed that the parents have a common age x and $\mu_{i,k}=\mu_{i,k}(t, x)$. Without giving any reason, the author states that $\partial\mu_{i,k}/\partial t$ are linear combinations of the $\mu_{i,k}$. This gives six ordinary differential equations (called partial for the occurrence of the parameter x). They are treated by the matrix method, formally as integral equations. In the case of a single race the author's assumptions would mean that the nativity $\mu(t, x)$ can change only exponentially.

W. Feller (Providence, R. I.).

Preinreich, Gabriel A. D. The theory of industrial replacement. Skand. Aktuarietidskr. **1939**, 1–9 (1939). [MF 119]

The author compares his own investigations [Econometrica **6**, 219–241 (1938)] with those of Lotka [Skand. Aktuarietidskr. **1933**, 51–63] on the same subject and criticizes the latter particularly from the mathematical point of view. *W. Fenchel* (Copenhagen).

Preinreich, Gabriel A. D. The economic life of industrial equipment. Econometrica 8, 12–44 (1940). [MF 845]

This article deals with the problem of general "economic wearing out" of capital equipment, as contrasted with the purely physical wearing out. First a brief discussion of the theory of economic life of machines is given, as originated by J. S. Taylor and further developed by H. Hotelling. It is pointed out that this theory applies mainly to a static economic situation. The author undertakes to build up a generalized theory taking in account dynamic developments. For this purpose, among others, the idea of an infinite chain of replacements is introduced. *A. Wald.*

Amoroso, L. The transformation of value in the productive process. Econometrica 8, 1–11 (1940). [MF 844]

A system of differential equations of economic movement is derived on the basis of three fundamental principles. The first of these principles is called the differential principle and can be formulated as follows: Let $Z_\mu(t)dt$ be the quantity of the product $\mu(\mu=1, \cdots, m)$ manufactured in $[t, t+dt]$ and denote by $x_{rs}(t)dt$ $(r=1, \cdots, m;$ $s=1, \cdots, n)$ the quantity of the factor of production s employed in the manufacture of r in $[t, t+dt]$. The differential principle says that each one of the $Z_1, \cdots, Z_m$ depends on all the x_{rs} and on all of their derivatives with respect to time. The second principle corresponds to the classical principle of maximizing utility and the third principle consists of the hypothesis that the individuals operate under conditions of free competition. The analogy of the system of differential equations obtained to that of classical mechanics is discussed. In particular, cost is interpreted as potential energy and the value of production as kinetic energy. The transformation of value which occurs in the dynamic of the productive process is likened to the transformation of energy in a mechanical process. *A. Wald* (New York, N. Y.).

Koopmans, T. The degree of damping in business cycles. Econometrica 8, 79–89 (1940). [MF 846]

To describe damped vibration, the author takes a function Z_t as the real part of Ae^{zt}, where $z=x+iy$, and t is the time. The "structure" of a system is first supposed given by "structural coefficients" A_n in a linear difference equation

$$A_N Z_t + A_{N-1} Z_{t-1} + \cdots + A_n Z_{t-N+n} + \cdots + A_0 Z_{t-N} = 0.$$

The character of the movement is determined by the n roots of the corresponding characteristic equation. With D denoting the damping ratio and T the period, variational analysis is used to determine expressions for $\partial D/\partial A_n$ and $\partial T/\partial A_n$. As an illustration, J. Tinbergen's equation for the United States is solved for its four roots, two real and two imaginaries.

To generalize, the author passes from the simple linear algebraic operator to a real linear operator Λ such that $\Lambda(e^{zt}) = L(z)\cdot e^{zt}$, and again finds the same lag of a quarter-period. *E. L. Dodd* (Austin, Tex.).

Boehm, Carl. Die statistische Erforschung von Zusammenhängen. Chem. Fabrik 12, 433–437, 445–451 (1939). [MF 454]

Steller, E. T. Critical remarks on some important points in actuarial science. Verzekerings-Arch. 20, 101–130 (1939). (Dutch) [MF 811]

This is a continuation of a previous article in the same volume [these Rev. 1, 23 (1940)] and is concerned with mortality tables and with a critical survey of several methods of computing life annuities. *W. Feller.*

Rusting, F. H. A property concerning the computation of premiums with abnormal risks. Verzekerings-Arch. 20, 84–100 (1939). (Dutch) [MF 810]

Stern, Erich. Kursformeln für Anleihen mit gleichmässig gestaffeltem Zins. Verzekerings-Arch. 20, 131–146 (1939). [MF 812]

Polidori, Ciro. La nuova teoria della capitalizzazione e i problemi d'interesse semplice. Period. Mat. 19, 126–140 (1939). [MF 562]

A general survey of recent developments. *W. Feller* (Providence, R. I.).

GEOMETRY

***Prager, W. Descriptive geometry. II.** Publ. Univ. Istanbul no. 44, 1–80 (1940). (Turkish)

Kobold, F. Eine einfache Herleitung der Flächenverzerrung, des Vergrösserungsverhältnisses und der Azimutreduktionen bei der winkeltreuen Zylinderprojektion. Schweiz. Z. Vermessgswes. 38, 8–15 (1940). [MF 1049]

Leemans, J. Triples de points ayant le même barycentre. Mathesis 53, 223–227 (1939). [MF 853]

Vectors are used to establish formulas which permit the derivation of triads of points having the same center of gravity as a given triad. Particular results are applied to the solution of some problems in the geometry of the triangle. *N. A. Court* (Norman, Okla.).

Goormaghtigh, R. Sur les triangles de Poncelet. Mathesis 53, 269–276 (1939). [MF 857]

Triangles are considered inscribed in a circle and circumscribed about a conic. Various properties are known relating to some of the geometric elements of such triangles; a study of these properties by one and the same method permits us to state them with greater precision, and to generalize some of them [Introduction]. The method referred to consists of the use of complex coordinates. *N. A. Court.*

Rangaswami, K. The theory of the general contact circles of a triangle. J. Indian Math. Soc. 3, 254–261 (1939). [MF 1021]

The conics are considered which are confocal with the conics inscribed in a given triangle (infocal conics). It is shown that the reciprocal of an infocal conic Σ with respect to the Steiner ellipse of the triangle is the harmonic of a circle [in the sense of H. P. Baker, Principles of Geometry, vol. 2, p. 122], called the contact circle of the infocal conic Σ. For the inconics themselves the contact circle degenerates into the contact line. If a variable infocal conic is tangent to a fixed line, its contact circle is orthogonal to a fixed circle. The nine-point circle of a triangle is tangent to the contact circles of the infocal conics which pass through the Lemoine point of the triangle. Other analogous results are obtained.

The paper generalizes properties of pairs of isogonal points

of a triangle, and has marked analogies with a previous paper by the same author [Math. Student **5**, 93–98 (1938)].

N. A. Court (Norman, Okla.).

Patterson, B. C. The triangle: its deltoids and foliates. Amer. Math. Monthly **47**, 11–18 (1940). [MF 1151]

To illustrate the "economy of expression" afforded by the analytical methods of Morley's inversive geometry when applied to the geometry of the triangle, the author derives the equation of the envelope of the Simson line, both for the classical case and for Poncelet's generalization. The envelope is called, with Morley, a deltoid, instead of a tricusp. The author also derives the equations of the pedal curves (foliates) of some points with respect to the Simson line. The "pertinence" of the method is further shown in the case of curves obtained as the locus of a fixed point of a circle, when the circle revolves about its center with constant velocity, while its center moves, with constant velocity, on another circle. *N. A. Court.*

Thébault, V. Géométrie du triangle et du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 347–357 (1939). [MF 1204]

The first part of the paper considers the successive pedal triangles of a point M for a given triangle (T), that is, the pedal triangle (T_1) of M for (T), the pedal triangle (T_2) of M for (T_1), etc. Known results are rendered more precise, and new results added. The successive pedal triangles of two isogonal conjugate points are compared, and the special case is considered, when the isogonal points are equidistant from the circumcenter of the triangle.

The part devoted to the geometry of the tetrahedron is summed up by the author in the following way. Given a tetrahedron $A_1A_2A_3A_4$ inscribed in a sphere (O), center O, and a sphere (Σ), center A_5, consider the spheres (O_1), (O_2), (O_3), (O_4), centers O_1, O_2, O_3, O_4 orthogonal to (Σ) and passing, respectively, through the vertices $(A_2A_3A_4)$, $(A_3A_4A_1)$, $(A_4A_1A_2)$, $(A_1A_2A_3)$; (a) the points A_5, O have the same barycentric coordinates for the tetrahedrons $A_1A_2A_3A_4$, $O_1O_2O_3O_4$; (b) the volumes u_1, u_2, u_3, u_4, u_5 of the tetrahedrons $O_2O_3O_4O_5$, $O_3O_4O_5O_1$, $\cdots$, and the volumes V_1, V_2, V_3, V_4, V_5, of the tetrahedrons $A_2A_3A_4A_5$, $A_3A_4A_5A_1$, $\cdots$ satisfy the relation

$$u_1 : V_1 = \cdots = u_5 : V_5;$$

(c) the volumes U_1, U_2, $\cdots$ of the antipedal tetrahedrons of the points A_1, A_2, $\cdots$ for the tetrahedrons $A_2A_3A_4A_5$, $A_3A_4A_5A_1$, $\cdots$ satisfy the relation

$$U_1 : V_1 = \cdots = U_5 : V_5.$$

N. A. Court (Norman, Okla.).

Thébault, V. On a sphere connected with a tetrahedron. Gaz. Mat. **45**, 292–295 (1940). (Roumanian) [MF 1402]

Nehring, Otto und Zacharias, Max. Einige Sätze über ebene Vielecke. Jber. Deutsch. Math. Verein. **49**, 123–134 (1939). [MF 680]

Fortsetzung einer Arbeit von Nehring [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1938**, 11–16]. Einem n-Eck A_i sei ein zweites n-Eck C_i derart umschrieben, dass C_iC_{i+1} durch A_i geht. Schneiden sich die Kreise (A_{i-1}, A_i, C_i) mit den Mittelpunkten M_i in einem Punkte S, so sind die n-Ecke M_i and C_i einander gleichsinnig ähnlich. S ist der Punkt, der bei der bezüglichen Ähnlichkeitstransformation sich selbst entspricht. Erweiterungen und Umkehrungen des Satzes. Besondere Fälle: A_i ist einem Kreise einbeschrieben; A_i ist ein Kreis $2n$-Eck mit parallelen Gegenseiten; A_i ist ein regelmässiges $2n$-Eck. *O. Bottema* (Deventer).

Auluck, F. C. On Poncelet polygons. Proc. Indian Acad. Sci., Sect. A. **10**, 342–343 (1939). [MF 909]

If the radii R, r and the segment d joining the centers of two circles satisfy the relation

$$\left[1-\left(\frac{r}{R-d}\right)^2\right]^{\frac{1}{2}}+\left[1-\left(\frac{r}{R+d}\right)^2\right]^{\frac{1}{2}}=1,$$

it is possible to inscribe a hexagon in the first circle that would be circumscribed about the other. An analogous relation is given for the pentagon. *N. A. Court.*

Jessen, Børge. On the volume of polyhedra. Mat. Tidsskr. A. **1939**, 35–44 (1939). (Danish) [MF 1057]

This is a lecture giving a new proof of a theorem by Dehn to the effect that it is impossible to divide a regular tetrahedron into a finite number of parts which would, when put together in a suitable manner, form a rectangular parallelepiped. *W. Feller* (Providence, R. I.).

Merz, K. Einseitige Ergänzungsvielflache aus dem Tetraedertyp. Vierteljschr. Naturforsch. Ges. Zürich **84**, 1–8 (1939). [MF 198]

Drei Ebenen, die durch einen Punkt im Innern eines konvexen Vielflachs gelegt werden, zerlegen dieses in zwei einseitige Vielflache. Ein Beispiel ist die Zerlegung des Tetraeders in Heptaeder und Pentadekaeder. Erweiterung dieses Beispiels: Zerlegung einer unendlichen Reihe $4n$-Flache in einseitige $(4n+3)$-Flache, sowie in $(8n+7)$-Flache (n ungerade), bzw. $(8n+3)$-Flache (n gerade).

O. Bottema (Deventer).

Merz, K. Kreuzhauben aus dem Oktaedertyp. Vierteljschr. Naturforsch. Ges. Zürich **84**, 137–144 (1939). [MF 199]

Die drei Quadrate der Achsenschnitte eines Oktaeders werden durch regelmässige $4m$-Ecke ersetzt. Für $m=2t+1$ und $m=2t$ entsteht ein $(24t+8)$-Flach. Weitere Konstruktionen anlässlich dieser Figur. *O. Bottema* (Deventer).

Vahlen, Th. Körperinhalt, nichteuklidisch. Monatsh. Math. Phys. **48**, 419–425 (1939). [MF 658]

Verfasser leitet Formeln ab für den Inhalt verschiedener Körper eines sphärischen Raumes. Das bekannte Ergebnis, wonach der Inhalt eines sphärischen Dreiecks seinem sphärischen Exzesse gleich ist, wird für Körper höherer Dimension verallgemeinert, und der Inhalt einer Kugelzweieckzone, eines Kugelzweiecks, einer Kugel und eines Kugelsektors, des sphärischen Raumes werden berechnet. Wenn ein Polygon mit den Seiten a, b, c, $\cdots$ sich in einem sphärischen Raum um $d\varphi$ um eine Achse dreht, die mit a, b, c, $\cdots$ die Winkel α, β, γ, $\cdots$ bildet, so entsteht der Rauminhalt $(d\varphi/2)\cdot\sum a\cdot\cos\alpha$. Von einem Polyeder werde die Ebene einer Seiten, fläche mit den Seiten a, b, $\cdots$ um einen Winkel $d\varphi$ gedreht, während die übrigen Seitenflächen festgehalten werden. Dann ist die Änderung des Winkels h an der Kante a gleich $\cos\alpha\cdot d\varphi$ und die Änderung des Polyeder inhalts ist $\frac{1}{2}\sum a\cdot dh$. Es seien a, b, c, d vier Punkte auf der K_3, $x^2+y^2+z^2+t^2=1$, a', b', c', d' die Ecken des Polar-Vierflachs; dann ist $abcd+a'b'c'd'=-\frac{1}{2}\sum ab\cdot c'd'+\pi^2$.

G. Schaake (Groningen).

Weitzenböck, R. Zur Theorie der Komplexgrössen a_{ijk}. Monatsh. Math. Phys. **48**, 129–140 (1939). [MF 632]

Plane coordinates p_{ijk} exist in $n-1$ dimensions, analogous to Plücker coordinates p_{ij} of straight lines in [3]. The p_{ijk} are the $C_{n,3}$ three-rowed minors of an array of three rows and n columns. More generally, coordinates a_{ijk} of a complex in $[n-1]$ exist; and when they satisfy certain quadratic relations (p-relations) they become specialized and represent planes (two dimensional linear spaces). The general complex a_{ijk} can be resolved into the sum of a finite number (s) of terms $\lambda_{ijk}p_{ijk}$: briefly into the sum of s planes. The author finds an upper limit to the value of s, namely $s \leq [((n-3)/2)^2+1]$, where $[r]$ means that greatest integer in the real number r. He points out that when $n=5, 6, 7, 8$, s actually attains this equality, and cites recent work by Reichel [Dissertation, Greifswald, 1907, p. 59], Schouten [Rend. Circ. Mat. Palermo **55** (1931)] and Gourewitsch [C. R. Acad. Sci. URSS **1934**, 567–569]. The precise result for $n>8$ awaits solution.

The method of work is an interesting combination of algebra and projective geometry. These new results for p_{ijk} depend upon the known theory of the p_{ij} in $[n-1]$, which is here sketched from several points of view, including the Grassmannian, the canonical matrix of a skew symmetric matrix and the invariant theory of complexes.

H. W. Turnbull (St. Andrews).

Mehmke, R. Zur metrischen Geometrie quadratischer Gebilde. Math. Ann. **117**, 1–16 (1939). [MF 1056]

The author continues his investigation [see these Rev. 1, 25 (1940)] of orthogonal projections on lines of a plane, and on lines and planes in space, using Grassmann's calculus of points. The following theorem on lines of a plane is typical. Let the orthogonal projections of a variable directed segment on n fixed lines represent forces acting on a rigid lamina. Let the magnitudes of these forces be respectively multiplied by n fixed numbers. Then the resultant is a force whose line of action passes through a fixed point.

H. S. M. Coxeter (Toronto, Ont.).

Egerváry, E. On orthocentric simplexes. Acta Litt. Sci. Szeged **9**, 218–226 (1940). [MF 1225]

The author proves the three propositions: (1) If the set $P_0P_1 \cdots P_{n+1}$ of $n+2$ points in the n-dimensional space is orthocentric, then and only then can the mutual distances $\overline{P_iP_j}$ be expressed by $n+2$ symmetric parameters λ_i in the form: $\overline{P_iP_j}^2=\lambda_i+\lambda_j$ ($i, j=0, 1, \cdots, n+1; i\neq j$), the parameters λ_i being restricted only by the relations

$$\frac{1}{\lambda_0}+\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_{n+1}}=0, \quad \lambda_i+\lambda_j>0 \text{ for } i\neq j.$$

(2) The $n+1$ points $P_i(\epsilon_{i1}, \epsilon_{i2}, \cdots, \epsilon_{i\,n+1})$ ($i=1, 2, \cdots, n+1$) in the n-dimensional space, whose homogeneous cartesian coordinates ϵ_{ij} are the elements of an orthogonal matrix, form together with the origin an orthocentric set of $n+2$ points. And conversely, if the "interior" point of an orthocentric set will be placed at the origin, then an orthogonal matrix ϵ_{ij} can be found, by means of which the coordinates of the other points are expressed in the form: $x_{ij}=\rho(\epsilon_{ij}/\epsilon_{i\,n+1})$ ($i=1, 2, \cdots, n+1; j=1, 2, \cdots, n$). (3) The orthocenters and the barycenters of all the $(k-1)$-dimensional simplexes of an orthocentric simplex of n dimensions (specified by the parameters $\lambda_1, \cdots, \lambda_{n+1}$) belong to a hypersphere S_{k-1} of radius $(1/2k)((n+1-2k)^2\lambda_0+\lambda_1+\lambda_2+\cdots+\lambda_{n+1})^{\frac{1}{2}}$, $k=1, 2, \cdots, n$. The middle points C_{k-1} of these hyperspheres S_{k-1} lie on the "Euler line" which joins the orthocenter P_0 and the barycenter G of the simplex, and divide the distance GP_0 in the ratio

$$\frac{\overline{GC_{k-1}}}{\overline{P_0C_{k-1}}}=\frac{n+1-2k}{n+1}, \qquad k=1, 2, \cdots, n.$$

N. A. Court (Norman, Okla.).

Gambier, Bertrand. Cercles perpendiculaires et un paradoxe relatif aux imaginaires. Bull. Sci. Math. **63**, 233–238 (1939). [MF 737]

Kárteszi, Ferenc. Anwendungen einer geometrischen Abbildung. Mat. Fiz. Lapok **46**, 146–151 (1939). (Hungarian) [MF 986]

Using a space curve of third order a one-to-one correspondence is defined between the totality of conic sections in a plane and the totality of the linear complexes in a space. As an application a new proof is given for a theorem of Poncelet. *O. Szász* (Cincinnati, Ohio).

Schatz, Heinrich. Kreisscharen mit konstanten Invarianten in der Geometrie von Laguerre. Jber. Deutsch. Math. Verein. **49**, 134–140 (1939). [MF 681]

In einer früheren Arbeit [Monatsh. Math. Phys. 1928] hat der Verfasser eine Methode angegeben, zwei reelle Invarianten ρ, τ der allgemeinen Zykelscharen unter der Laguerreschen Transformationsgruppe in der Ebene zu definieren. Jetzt behandelt der Verfasser das Problem der Bestimmung aller Zykelscharen mit zwei konstanten reellen Invarianten unter der Laguerreschen Gruppe. Das Problem lässt sich durch isotrope Projektion auf das Problem der Bestimmung der Raumkurve mit zwei konstanten Bewegungsinvarianten, das heiszt, mit konstanter Krümmung und konstanter Torsion zurückführen, wobei die Massbestimmung durch

$$\pm ds^2=dx_1^2+dx_2^2+dx_3^2$$

definiert wird. Hierbei bedeuten x_1, x_2 die rechtwinkligen Koordinaten des Zentrums des Zykels und x_3 den Radius des Zykels.

Zuerst betrachtet der Verfasser den Fall, dass $\dot{x}_1^2+\dot{x}_2^2-\dot{x}_3^2\neq 0$ ($\dot{x}_i=dx_i/dt$) ist, und dieser Fall wird nach dem Vorzeichen von

$$\left(\frac{dx_1}{ds}\right)^2+\left(\frac{dx_2}{ds}\right)^2-\left(\frac{dx_3}{ds}\right)^2$$

und von

$$\left(\frac{d^2x_1}{ds^2}\right)^2+\left(\frac{d^2x_2}{ds^2}\right)^2-\left(\frac{d^2x_3}{ds^2}\right)^2$$

in fünf Fälle geteilt; der Fall $\dot{x}_1^2+\dot{x}_2^2-\dot{x}_3^2=0$ wird in zwei Fälle geteilt. Das Problem wird für diese Fälle rechnerisch gelöst und es erweist sich, dass die Zykelscharen ($\bar{W}$-Scharen) eingliedrige Gruppen von Laguerre-Transformationen gestatten. Von diesem Standpunkte aus hat der Verfasser infinitesimale Transformationen der eingliedrigen Gruppe die diese Zykelscharen gestatten bestimmt.

T. Kubota (Sendai).

Fejes, László. Über den Schmiegungspolyeder. Mat. Fiz. Lapok **46**, 141–145 (1939). (Hungarian) [MF 985]

Regarding the definition of the osculating polyhedron of a convex body, we refer to the previous paper of the author [Mat. Fiz. Lapok **45**, 199 (1938)]. Let P be a convex polyhedron, E one of its vertices. In the present paper the

necessary and sufficient condition is obtained for the existence of a convex body K containing E in its interior such that the osculating polyhedron of K is identical with P. The condition is that none of the faces of P meeting at E should be a triangle. *G. Szegö.*

Neumann, B. H. On some affine invariants of closed convex regions. J. London Math. Soc. **14**, 262–272 (1939). [MF 431]

Let C be a plane closed convex curve, let X be a point inside C, and let PQ be an oriented chord through X. The ratio $\rho=PX/PQ$ assumes a positive minimum for X fixed and P varying on C. For X varying inside C, this minimum assumes a maximum $\mu_1=\max_{(X)}\min_{(P)}\rho$ which is an affine invariant of C. The author proves

$$(*)\qquad \tfrac{1}{3}\leqq\mu_1\leqq\tfrac{1}{2},$$

$\mu_1=\frac{1}{3}$ for triangles and $\mu_1=\frac{1}{2}$ if C has a centre, and only in these cases. This is closely related to the following problem considered by Favard [for reference, see, for instance, Bonnesen-Fenchel: Theorie der konvexen Körper, Ergebnisse der Mathematik, Berlin, 1934, pp. 86–87]: Let C' be the curve symmetric to C with respect to an arbitrary point. To find the smallest curve homothetic to C' and circumscribed to C. By means of results concerning this problem (obtained independently), the proof of (*) is reduced to that of similar inequalities for an analogous invariant defined with 3 chords through X. These latter inequalities are easily obtained by means of an elementary theorem on triangles. Also the exact bounds of invariants μ_{2n+1} defined with an arbitrary odd number of chords are found by induction. *W. Fenchel* (Copenhagen).

Hornich, Hans. Eine allgemeine Ungleichung für Kurven. Monatsh. Math. Phys. **47**, 432–438 (1939). [MF 72]

Let C be a rectifiable plane curve with length L. Consider the set $M(d, C)$ of all points with distance not greater than d from C. Let $F(d, C)$ be the area of $M(d, C)$; then

$$F(d, C)\leqq\pi d^2+2dL,$$

and if a positive value of d exists for which equality is valid, C has a continuous varying tangent, and any circle through three points of C has a radius not smaller than this d. The inequality follows immediately from the corresponding elementary inequality for approximating polygons. The proof of the second part is based on the following remark: if there exists such a d for C, this d has the same property for any curve which is a part of C. This shows that the assumption made implies local properties of C. An inverse theorem is also obtained. *W. Fenchel* (Copenhagen).

Bol, G. Ein isoperimetrisches Problem. Nieuw Arch. Wiskde **20**, 171–175 (1940). [MF 1094]

Let C denote a convex curve with "corners," the angle between the two extreme tangents at a corner being ψ_i. The following generalization of the isoperimetric property of the circle is proved: Among all curves C with given angles ψ_i and given perimeter those and only those have greatest area that are obtained from a circle by replacing arcs of the circle by pairs of tangents including the angles ψ_i. The proof is based on Minkowski's inequality for mixed areas: $F_{12}^2-F_{11}F_{22}\geqq 0$. As a special case it follows that among all polygons with given angles and perimeter those circumscribed to a circle have greatest area. *F. John.*

Minoda, Takashi. Some theorems on convex polygons and ovals. Tôhoku Math. J. **46**, 106–116 (1939). [MF 1173]

Minoda, Takashi. A supplement to "Some theorems on the convex polygons and ovals." Tôhoku Math. J. **46**, 159 (1939). [MF 1179]

The author derives necessary and sufficient conditions for a convex polygon P to be such that every secant through a fixed interior point O divides the perimeter and area of P in the same ratio. (The author's formulation of Theorem 1 does not express exactly what is proved.) Examples of sufficient conditions are that P has O as center, or that P is circumscribed to a circle about O; the second condition is also necessary if P has no equal or parallel sides. Conditions are also derived for the existence of a point O such that every pair of half lines from O including a fixed angle θ divides perimeter and area of P in the same ratio. Analogous results are obtained for general convex curves. *F. John* (Lexington, Ky.).

Fiala, Félix. Une inégalité isopérimétrique sur les surfaces ouvertes à courbure positive. C. R. Acad. Sci. Paris **209**, 821–823 (1939). [MF 858]

Announcement of results related to the generalization of the isoperimetric problem to two-dimensional (normal) manifolds M, homeomorphic to the plane and provided with a Riemannian metric of nonnegative curvature. For the total curvature C of M, Cohn-Vossen's result $C\leqq 2\pi$ holds. An isoperimetric inequality between the length L of a curve on M and its area F is $L^2\geqq 2F(2\pi-C)$. For $C=2\pi$ there exists a bound $L^*>0$, depending on M, such that F is certainly bounded for each $L<L^*$; but the limit inferior of L^2/F equals $2(2\pi-C)$. The total area of M is infinite. *H. Lewy* (Berkeley, Calif.).

MATHEMATICAL PHYSICS

Kemble, Edwin C. Fluctuations, thermodynamic equilibrium and entropy. Phys. Rev. **56**, 1013–1023 (1939). [MF 482]

Zur Untersuchungen der Schwankungsphänomene in der Formulierung der thermodynamischen Sätze schlägt der Verfasser vor, alle Aussagen der Thermodynamik über individuelle Systeme zu ersetzen durch entsprechende Aussagen über Gesamtheiten von identisch konstruierten, ähnlich präparierten und denselben experimentellen Operationen unterworfenen Systemen. Indem die thermodynamischen Aussagen über den Carnot'schen Kreisprozess nur im statistischen Mittel über viele ähnliche solche Prozesse aufrecht erhalten werden (mit der Ergänzung, dass die Schwankungen in solchen Ensembles mit wachsender Masse des Systems im thermischen Gleichgewicht nach Null konvergieren), wird der Begriff der Gibbs'schen statistischen Gesamtheit in die phänomenologische Thermodynamik eingeführt. Der Verfasser unterscheidet zwei verschiedene Operationen, um ein thermodynamisches Gleichgewicht herzustellen. Bei der Operation α wird das System isoliert und seinen Teilen wird es ermöglicht, Energie auszutauschen, bis ein Gleichgewicht hergestellt ist. Bei der Operation β wird das gegebene System A während langer Zeit in thermischen Kontakt mit einem Reservoir B, dem "Wärmebad," gebracht, dessen

Wärmekapazität gross ist und dessen Temperatur, die durch ein Thermometer beobachtet werden kann (äquivalent einer approximativ-makroskopischen Energiemessung von B), gleichförmig und konstant ist. Die Thermodynamik braucht die Behauptung, dass, wenn nur die Energie von A vernachlässigbar ist gegen die von B, durch die Operation β ein durch die Temperatur von B eindeutig bestimmter Zustand von A mit bestimmter Energieverteilung erzeugt wird. In Bezug auf das System $A+B$ ist die Operation β eine Spezialisierung der Operation α, indem das isolierte Gesamtsystem $A+B$ hier insbesondere in zwei Teilsysteme geteilt ist, von denen das eine viel mehr Freiheitsgrade hat als das andere, während in Bezug auf das Teilsystem A die Operation β zu neuen Aussagen führt. Die Operation α entspricht der mikrokanonischen Gesamtheit von Boltzmann, die Operation β der kanonischen Gesamtheit von Gibbs. Der Verfasser gibt aus verschiedenen Gründen [vgl. das folgende Referat] der Operation β gegenüber der Operation α den Vorzug und stellt sich die Aufgabe, die bisher befriedigendste Herleitung der Grundannahmen der statistischen Wärmetheorie aus der Quantenmechanik, die von v. Neumann [Z. Phys. **57**, 30 (1929)] herrührt, so zu modifizieren, dass statt der Operation α gleich die zur kanonischen Gesamtheit führende Operation β verwendet wird. Der Fortschritt der Neumann'schen Formulierung bestand darin, dass keine Annahmen über Gleichwahrscheinlichkeit gewisser Anfangszustände und über ungeordnete Phasenwerte des Systems eingeführt werden mussten. Statt dessen spielte der Begriff des makroskopischen Beobachters eine wesentliche Rolle, von dem angenommen wurde, dass er nur einen Teil der Observablen des Systems, nämlich die nach Voraussetzung miteinander vertauschbaren makroskopischen Grössen beobachten kann. Zu diesen Grössen gehört nach v. Neumann speziell die makroskopische Energie, von der angenommen wird, sie sei zeitlich konstant, das heisst, mit dem exakten Hamiltonoperator des Systems (mikroskopische Energie) vertauschbar. An Stelle der Unordnungsannahmen tritt bei v. Neumann der Begriff der "Wahrscheinlichkeit eines makroskopischen Beobachters" bezw. der Mittelung über die Resultate aller möglichen makroskopischen Beobachter. Unter allgemeinen Annahmen hat v. Neumann bewiesen, dass die Zeitgesamtheit eines isolierten Systems für "fast alle" makroskopischen Beobachter mit der mikrokanonischen Gesamtheit übereinstimmt [vgl. auch das folgende Referat]. Der Verfasser kann den erwähnten Vorteil der v. Neumann'schen Formulierung der Theorie beibehalten und überdies für das Teilsystem A die mikroskopische Definition der Entropie ($S=-k$ spur $\rho \log \rho$, $k=$Gaskonstante, $\rho=$Dichtematrix) benützen, indem er zwar nicht das ganze System $A+B$, wohl aber das Teilsystem A als durch den thermodynamischen Beobachter exakt messbar annimmt. An Stelle der Neumann'schen Mittelung über die Ergebnisse aller möglichen makroskopischen Beobachter tritt beim Verfasser die Mittelung über alle physikalischen möglichen Wechselwirkungen zwischen A und B.

Mathematisch wird dies Problem folgendermassen formuliert: Nach Herstellung des thermischen Kontaktes zwischen den Systemen A und B sei die Hamiltonfunktion des Gesamtsystems $H=H_A+H_B+H'$, worin H_A, H_B die ursprünglichen Hamiltonfunktionen der separierten Teilsysteme und H' die von den Koordinaten beider Systeme abhängige Wechselwirkungsenergie ist. Der Effekt der Wechselwirkung auf die statistische Dichtematrix ρ_A (deren Resultat symbolisch mit β_{ρ_A} bezeichnet wird) ist bestimmt durch die unitäre Matrix U der kanonischen Transformation, die von einem Schema mit diagonaler Matrix H_A+H_B zum neuen Schema mit diagonalem H_A+H_B+H' führt. In Analogie zur Behandlungsweise von v. Neumann bildet der Verfasser nun erstens den Zeitmittelwert von β_{ρ_A}, zweitens aber den statistischen Mittelwert über alle möglichen U, die mit der Annahme verträglich sind, dass H' klein sein soll gegen H_A und H_B. Der letztere Mittelwert ist eindeutig durch die Gruppentheorie definiert. Der Verfasser kommt zu seinem Hauptresultat, dass in einem Schema, wo H_A diagonal ist, das Resultat dieses Mittelungsprozesses gegeben ist durch

$$\overline{(\beta_{\rho_A})}_{U,\,t}(n, n')=\delta_{nn'}e^{-\gamma E_{A_n}}/\sum_n e^{-\gamma E_{A_n}}$$

(E_{A_n}=Energiewerte von A, γ reine Temperaturfunktion von B, die mit $1/kT$ identifiziert werden kann), was der kanonischen Gesamtheit entspricht. Details des Beweises werden in der nachfolgenden referierten Arbeit gegeben. Die Arbeit schliesst mit einer Diskussion bekannter Extremaleigenschaften der mikroskopischen Entropie, besonders des Gibbs'schen Satzes nach welchem im thermischen Kontakt die Summe der Entropien der Teilsysteme stets grösser ist als zur zeit der Herstellung des Kontaktes. Dieses Theorem wird auch herangezogen bei der statistischen Herleitung des zweiten Hauptsatzes. Im Falle adiabatischer Prozesse wird die Notwendigkeit eines besonderen "adiabatischen Calorimeters" hervorgehoben, das dafür zu sorgen hat, dass das System wirklich eine Folge thermodynamischer Gleichgewichtszustände durchläuft. *W. Pauli* (Zürich).

Kemble, Edwin C. The quantum-mechanical basis of statistical mechanics. Phys. Rev. **56**, 1146–1164 (1939). [MF 698]

Der Verfasser beginnt mit einer kurzen Uebersicht über den jetzigen Stand der quantenmechanischen Grundlagen der statistischen Mechanik. Nach früheren Arbeiten [W. Pauli, Sommerfeld Festschrift, Leipzig, 1928, p. 30], die noch spezielle Annahmen über einen ungeordneten Charakter des Anfangszustandes verwendeten, erreichte v. Neumann [J. v. Neumann, Z. Phys. **57**, 30 (1929). Vgl. auch W. Pauli und M. Fierz, Z. Phys. **106**, 572 (1938)] einen gewissen Abschluss. Er führte miteinander vertauschbare makroskopische Observable ein, zu denen auch eine makroskopische Energie gehört. Zum gleichen Wert aller makroskopischen Variablen gehört eine Gruppe von $S_{v,a}$ Eigenfunktionen des isoliert gedachten Systems, von denen man sagt, sie definieren eine bestimmte "Makrozelle." Die Doppelindizierung besagt, dass die Zellen mit gleichen a zum selben Wert der Makroenergie gehören, während v von 1 bis N_a läuft, wo N_a die Zahl der Makrozellen zum gegebenen Wert der Makroenergie, das heisst, zur selben "Energieschale" bedeutet und $G_a=\sum_{v=1}^{N_a} S_{v,a}$ das Gesamtgewicht der Energieschale ist. Von der Makroenergie wird vorausgesetzt, sie sei zeitlich konstant, so dass die Matrix U, die von einem Schema mit diagonaler Mikroenergie des Systems zu einem solchen mit diagonalen Matrices der makroskopischen Variablen führt, in endliche Kästchen von je G_a Dimensionen zerfällt. Sei nun ψ_t ein reiner Mikrozustand des Systems und $x_{v,a}(t)$ die Wahrscheinlichkeit, das System hierbei in der Makrozelle v, a anzutreffen. Dann definiert v. Neumann die "makroskopische Entropie" durch

$$S_{\mathrm{mac}}(\psi_t)=-k\sum_{a,v} x_{v,a}(t) \log x_{v,a}(t)/S_{v,a}. \tag{1}$$

Ist $u_a=\sum_v x_{v,a}(t)$ die (zeitlich konstante) Wahrscheinlichkeit, das System in der Energieschale a anzutreffen, wobei $\sum_a u_a=1$, so ist die zugehörige "uniformisierte Gesamtheit"

durch $\bar{x}_{\nu,a}=u_a S_{\nu,a}/G_a$ definiert mit der zeitlich konstanten Entropie

$$\text{(2)} \qquad \bar{S}_{\text{mac}}(\psi_t)=\bar{S}_{\text{mac}}(\psi_0)=-k\sum_a u_a \log (u_a/G_a),$$

und es ist leicht zu sagen, dass $S_{\text{mac}}(\psi_t)\leqq\bar{S}_{\text{mac}}(\psi)$. Neumann beweist nun das "H-Theorem," nämlich,

$$\bar{S}_{\text{mac}}(\psi)-\overline{S_{\text{mac}}(\psi_t)}\ll\bar{S}_{\text{mac}}(\psi)$$

[$\overline{S_{\text{mac}}(\psi_t)}$ bedeutet den Zeitmittelwert von $S_{\text{mac}}(\psi_t)$] und das "Ergoden-Theorem," nämlich:

$$\text{Zeitmittel von } [\bar{a}_u-\bar{a}_t]^2\ll[\bar{a}_u]^2$$

($\bar{a}_t$ = Erwartungswert einer beliebigen makroskopischen Observablen α zur Zeit t, $\bar{a}_u$ = Erwartungswert von α für die uniformisierte Gesamtheit) und zwar unter folgenden Annahmen: (1) Der exakte Hamilton-operator ist nicht (oder nicht zu stark) degeneriert. (2) Die Theoreme gelten zwar nicht für alle makroskopischen Beobachter, wohl aber bei gegebenen Werten von $S_{\nu,a}$ und N_a, für die erdrückende Mehrzahl derselben, wenn nur $G_a\gg N_a$ ist.

Der Verfasser macht jedoch Einwände gegen die Identifizierung der makroskopischen Entropie (1) mit der thermodynamischen. (a) Es scheint ihm unbefriedigend, dass die makroskopischen Observablen alle exakt vertauschbar seien. (b) Um dem Nernst'schen Theorem Rechnung zu tragen, wünscht er die Entropiedefinition (2) auch anzuwenden auf den Fall dass $N_a=1$ und G_a klein ist (im limes $T\to 0$ sogar $G_a=1$), während v. Neumann's Theoreme $G_a\gg 1$ verlangen. (c) Wie in der obenstehend referierten Arbeit ausgeführt, zieht der Verfasser die Operation β der Operation α vor. (d) Die makroskopische Entropiedefinition ist nicht auf mikroskopische Systeme mit nur wenigen Freiheitsgraden anwendbar. Der Verfasser bespricht sodann die Eigenschaften der mikroskopischen Entropiedefinition $S_{\text{mic}}=-k$ spur $\rho \log \rho$ (ρ = Dichtematrix), auf welche die obigen Einwände nicht zutreffen. Um sie sinnvoll anzuwenden, muss man jedoch das untersuchte System nicht als isoliert betrachten, sondern als Teil eines Gesamtsystems $A+B$, wobei A und B sich in thermischem Kontakt befinden. Aus der Dichtematrix $\rho_{A+B}(nm, n'm')$ des Gesamtsystems, wobei n die Zustände von A, m die von B nummeriert, erhält man durch Bildung der Partialspuren die Dichtematrizes

$$\rho_A(n, n')=\sum_m \rho_{A+B}(nm, n'm); \quad \rho_B(m, m')=\sum_n \rho_{A+B}(nm, nm')$$

der Teilsysteme. Die mikroskopische Entropie des Gesamtsystems ist nach einem Theorem von Gibbs [J. W. Gibbs, Elementary Principles of Statistical Mechanics, chap. 11, Th. 7], das von O. Klein [Z. Phys. **72**, 767 (1931)] in die Quantenmechanik übertragen wurde, kleiner als die Summe der Entropie der Teilsysteme, und gleich dieser Summe nur dann, wenn speziell $\rho_{A+B}=\rho_A\cdot\rho_B$ war. [Bei Gibbs sind im Sinne der klassischen Mechanik an Stelle der Dichtematrizes gewöhnliche Dichtefunktionen P (p, q) im Phasenraum verwendet und $S=-k\int P \log P d\omega$ ($d\omega$ = Volumelement des Phasenraumes) gesetzt. Er bewies $S_{A+B}\leqq S_A+S_B$ mit Gültigkeit des Gleichheitszeichens nur für $P=P_A\cdot P_B$, während allgemein $P_A=\int P d\omega_B$, $P_B=\int P d\omega_A$.] Während für das Gesamtsystem die mikroskopische Entropie stets zeitlich konstant bleibt, ist das nicht der Fall für S_A+S_B. Um den zeitlichen Verlauf dieser Summe zu berechnen, reicht das in Rede stehende Theorem nicht aus. Man kann aus ihm nur schliessen, dass bei Unabhängigkeit der Teilsysteme im Zeitpunkt t_0 infolge der Wechselwirkung zwischen A und B im unmittelbar folgenden Zeitpunkt $t_0+\Delta t$ (praktisch stets) gelten muss:

$$S_A(t_0+\Delta t)+S_B(t_0+\Delta t)>S_A(t_0)+S_B(t_0).$$

Der Verfasser will diesen Sachverhalt als vollen Ersatz des Prinzips des Anwachsens der Entropie infolge irreversibler Prozesse auffassen. Das wesentliche Resultat der Arbeit (dessen Beweis der Verfasser selbst nicht für mathematisch streng hält) besagt, dass das System A bei beliebigem Anfangszustand durch hinreichend langen Kontakt mit dem Thermostat B in eine kanonische Gesamtheit übergeführt wird, wenn nur die Wärmekapazität von B als hinreichend gross vorausgesetzt wird. Seien H_A, H_B die Hamiltonoperatoren von A bezw. B, H' die Wechselwirkung und $H=H_A+H_B+H'$. Seien ferner E_{A_n}, E_{B_m}, W_k bezw. die Eigenwerte von H_A, H_B und H, so hängt das Resultat der "Operation β" [siehe vorstehendes Referat] wesentlich ab von der Matrix $U(nm, k)$, die von einem Schema mit diagonalem H_A und H_B zu einem solchen mit diagonalem H führt. Der Natur der Sache nach muss nun angenommen werden: (a$_1$) das System B ist ein makroskopisches mit einem Abstand der Energieniveaus, der klein ist gegen alle anderen betrachteten Energieintervalle; (a$_2$) jedes System B hat eine bestimmte Temperatur, die aus dem Erwartungswert $\bar{E}_B$ der Energie der Systeme B zu erschliessen ist; (a$_3$) die Energie der System A sei bekannt und klein gegen E_B, so dass zwischen $\bar{E}_B$ bezw. der Streuung ΔE_B der Energie von B und den Grössen $\bar{W}$ bezw. ΔW für das Gesamtsystem nicht unterschieden werden muss; (a$_4$) die Erwartungswerte

$$e_k=W_k-\sum_n\sum_m E_{nm}|U(nm, k)|^2$$

von H' in den Eigenzuständen von H und die entsprechenden von $(\exp((H'-e_k)/\Delta W)^2-1)$ sind für alle k beschränkt durch

$$\bar{e}-e'<W_k-\sum_n\sum_m E_{nm}|U(nm, k)|^2<\bar{e}+e', \qquad e'\ll\bar{e},$$

und, für $p\ll P\ll 1$,

$$P-p<\sum_n\sum_m\left\{\exp\left[\left(\frac{W_k-e_k-E_{nm}}{\Delta W}\right)^2\right]-1\right\}|U(nm, k)|^2<P+p.$$

Diese Ungleichungen sind der Ersatz für die Zerfällung von U in Kästchen bei Neumann. Sie ermöglichen die Bildung eines Mittelwertes der Dichtematrix ρ_A des Teilsystems A nicht nur über die Zeit t, sondern auch (gruppentheoretisch) über alle mit diesen Ungleichungen verträglichen unitären Matrizes U. Indem ein angehängter Index U bezw. t den gruppentheoretischen Mittelwert über alle diese U bezw. den zeitlichen Mittelwert über lange Zeiten bezeichnet, zeigt der Verfasser, dass bei hinreichend grosser Wärmekapazität von B

$$[\rho_A(n, n')]_{U,t}=\delta_{nn'}\exp[(\psi_A-E_n)|kt],$$

und für die Schwankungen

$$[|\rho_A(n, n')-[\rho_A(n, n')]_{U,t}|^2]_{U,t}\ll|[\rho_A(n, n)]_{U,t}|^2.$$

Hierbei muss ganz analog zur Voraussetzung v. Neumann's Fehlen von Entartung der Energiewerte von H angenommen werden, nämlich

$$W_{\bar{k}}-W_k\neq W_{\bar{k}'}-W_{k'}$$

ausser für $\bar{k}=k$, $\bar{k}'=k'$ oder für $\bar{k}=\bar{k}'$, $k=k'$.

W. Pauli (Zürich).

AUTHOR INDEX

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